THREE TOPICS IN SET THEORY: FINITENESS AND CHOICE, CARDINALITY OF COMPACT SPACES, AND SINGULAR JÓNSSON CARDINALS

By

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I dedicate this work to my father and mother, Hugo, Oana, and my good friends.

—A mi parecer, este negocio en dos paletas le declararé yo, y es así: el tal hombre jura que va a morir en la horca, y si muere en ella, juró verdad, y por la ley puesta merece ser libre y que pase la puente; y si no le ahorcan, juró mentira, y por la misma ley merece que le ahorquen.

Cervantes, *Don Quijote de la Mancha.*

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THREE TOPICS IN SET THEORY:
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In this dissertation we work on three different topics in set theory.

The first topic deals with weak choice principles of the form: *Every "finite" family of non-empty sets has a choice function*, where "finite" stands for one of several different definitions of finiteness that are not equivalent unless we assume the axiom of choice (AC). Several relations of implication and independence are established.

In the second chapter we construct a model of set theory without AC where there exists a countable compact Hausdorff space with no isolated points, and a more elaborate model where there exists a countable compact Hausdorff space which is connected.

The last topic is the study of singular Jónsson cardinals using the techniques of core model theory. We extend a result of Mitchell to include singular cardinals.

INTRODUCTION

This work addresses several questions regarding three separate topics in set theory. The first two topics, which correspond to Chapters 1 and 2, deal with some of the unexpected phenomena that can arise when the axiom of choice (AC) is not assumed. The third topic is the study of some large cardinals of "medium size," in particular the so-called *Jónsson cardinals*, using the techniques of core model theory.

In the first chapter we study the regulating influence of AC and some of its weaker consequences on the notion of finiteness. When in the historical development of mathematics the use of actual infinities became progressively acceptable, the need to find precise definitions for the intuitive notions of finiteness and infinity became pressing. Among the early inquires into the subjects were those of Bernoulli and Dedekind, but until the development of set theory by Cantor there was no solid framework for the study of infinite quantities. The currently accepted definition of finiteness (which automatically provides a definition of infinity, by negation) is the following: a set is finite if it has n elements, for some positive integer number n.

One of the earliest systematic studies of finiteness was done by Tarski [Tar24]. One of his motivations was to find a purely set theoretical definition of a finite set, in order to avoid the appeal to the notion of number. After the development of the Von Neumann ordinals, which allowed mathematicians to consider numbers as sets (and therefore no longer an external notion), this motivation became less important. However, the many definitions that he included, along with the relations of implication between them that he found, plus the fact that all these definitions become equivalent under AC, turned out to be very influential in the realization of the importance of AC in resolving many fundamental matters in mathematics.

The main contribution of this chapter is the study of several weak versions of AC of the form *"every finite family of non-empty sets has a choice function."* If by "finite" we understand the commonly accepted definition, the statement above can be proved with the usual axioms of set theory (ZF), without any use of AC. But if we allow the use of less restrictive definitions of finiteness, the resulting statements cannot be proved in ZF unless some extra principle is assumed (usually a consequence of AC). Several different results are obtained, establishing relations of implication and independence between these statements and several well known choice principles.

In Chapter 2 we present some examples of the unusual behavior that can arise when AC fails, even in well studied areas of mathematics like topology.

Some of the best understood objects in topology are compact Hausdorff topological spaces. However, the rich structure of these spaces and the many properties that mathematicians have become used to expecting from them depend on AC. The particular property that we study here is the following:

Every compact Hausdorff space with no isolated points
has cardinality greater than or equal to that of the continuum. (1)

This can be proved easily using AC; however, we show here that this result cannot be proved from ZF alone, by providing a model of ZF where there exists a topology on ω which is compact Hausdorff and has no isolated points.

The result above answers a question posed by Marianne Morillon (see Miller [Mil93]). However, it can be proved that the example provided is an extremally disconnected space. In order to answer another question of Morillon (personal communication, 1998), the main theorem of the chapter produces an example of a countable compact Hausdorff space that is *connected.*

Both of the examples obtained in this chapter are compact spaces but lack a choice function that, given an open cover, chooses a particular finite subcover. This leads us to consider the notion of *effective compactness*, which requires the existence of such

a choice function. Section 2.2 is devoted to proving a result almost as good as (1) without using AC, but requiring the space to be effectively compact.

In Chapter 3 we study singular Jónsson cardinals using the tools of core model theory. The results that we obtain are an extension of results obtained by Kunen, Jensen, and Mitchell, which dealt mostly with regular Jónsson cardinals. The result presented here is an extension of the result by Mitchell [Mit99].

Jónsson cardinals are examples of large cardinals. Their consistency strength lies below that of measurable cardinals, but these cardinals are large enough for their existence to be incompatible with $V = L$ (that is, the affirmation that all sets are in Gödel's class of constructible sets). In terms of consistency strength, Jónsson cardinals are close to Ramsey and Rowbottom cardinals; however, the exact relations with those cardinals are not completely known yet. It can be proved, for example, that every Ramsey cardinal is also a Jónsson cardinal; however, it is not known (although it seems unlikely) whether every regular Jónsson cardinal is a Ramsey cardinal.

Core model theory offers many insights in this matter. In his most recent paper on the matter, Mitchell proves that if there is no inner model with a Woodin cardinal and the Steel core model K exists, then every regular Jónsson cardinal (in V) is a Ramsey cardinal in K; furthermore, if κ is a δ-Jónsson cardinal in V and δ is regular in V, then κ is a δ-Erdős cardinal in K. The same results are obtained in the absence of the Steel core model K if some extra assumptions are made.

In the same paper it is asked how can these results be generalized to include singular cardinals. Chapter 3 deals with one of the possible cases, namely, when δ is singular. However, in order to simplify the arguments, we will consider stronger hypotheses that will allow us to use the theory of the core model for sequences of measures; namely, we will assume that for every cardinal η, the order of η is less than η^{++}. This way we avoid the need to use iteration trees while at the same time we retain the essential features of the arguments. It seems plausible that the results obtained here

can also be obtained using hypotheses similar to those of Mitchell [Mit99], especially since the structure of the proof here follows closely the structure of the proof in that paper.

CHAPTER 1
FINITENESS AND THE AXIOM OF CHOICE

1.1 Introduction

It is easy to prove in ZF that every finite family of non-empty sets has a choice function, that is, a function that assigns to each set in the family one of its elements. The axiom of choice (AC) states that this holds also for infinite families, but it is well known that AC cannot be proved in ZF.

However, without AC, the notion of finiteness itself is not so clear, since different statements that express properties that we expect finite sets to have (and which are equivalent under AC) are not provably equivalent from ZF. From the several statements considered in the literature, in particular in Tarski [Tar24], Lévy [Lév58], and Howard and Rubin [How98], as possible definitions of finiteness, it is commonly agreed that the "right" definition is the most restrictive one: namely, equivalence with a natural number. It is this notion of finiteness for which the statement at the beginning is true.

In this chapter we study the relative strength of principles obtained by modifying the statement above to use less restrictive notions of finiteness. Let $\mathcal{Q}$ be a class of sets considered to be finite according to some notion of finiteness. Then the principle $C(\mathcal{Q})$ states that every family $F \in \mathcal{Q}$ of non-empty sets has a choice function. It happens that these principles are always independent from ZF, unless $\mathcal{Q}$ is the usual notion of finiteness mentioned above.

Another group of principles $C^{-}(\mathcal{Q})$ is studied, which states that for every infinite family $F \in \mathcal{Q}$ of sets there is an infinite subfamily $F' \subset F$ with a choice function. Many easy relations can be found between these statements and between these and some other well known choice principles; other not so trivial relations are established

in this chapter. It turns out that these interrelations are different for ZF and for ZFA, the set theory with atoms.

1.2 Notions of Finiteness

In this section we will give a list of well known notions or definitions of finiteness, and some of the relations between them. We follow the structure of note 94 from Howard and Rubin [How98], where some extra information can be found.

Degen [Deg94] introduces a reasonable formalization of the concept of a *notion of infinity*, which automatically provides a formal concept of a *notion of finiteness.* Suitably adapted, a Degen notion of finiteness is a formula $\phi(x)$ with one free variable such that the following are theorems of ZF:

1. $\forall \alpha \geq \omega \quad \neg\phi(\alpha)$,
2. $\forall n < \omega \quad \phi(n)$,
3. $\forall x \forall y$(if there exists a bijection $f\colon x \to y$ and $\phi(x)$ holds, then $\phi(y)$ holds),
4. $\forall x \forall y (x \subset y \wedge \phi(y) \to \phi(x))$.

However, our concept of a definition of finiteness is less restrictive. We will consider some notions (formulas) that do not satisfy (4), and one that does not satisfy (1). Since we do not actually need to give a formal definition of a notion of finiteness, it will be enough to remark that any such notion must be equivalent, under AC, to notion I (see below).

The first group of notions is taken from Lévy [Lév58]. Lévy introduced Ia and VII, while definitions I, II, III, and V were introduced by Tarski [Tar24], as well as VI (attributed to Tarski by Mostowski [Mos38]). Notion IV was originally introduced by Dedekind.

Definition 1. A set X is said to be

- *I-finite* if every non-empty family of subsets of X has a maximal element under inclusion.
- *Ia-finite* if it is not the disjoint union of two I-finite sets.

- *II-finite* if every non-empty family of subsets of X which is linearly ordered by inclusion has a maximal element (under inclusion).
- *III-finite* if there is no one-to-one map from $\mathcal{P}(X)$ into a proper subset of $\mathcal{P}(X)$.
- *IV-finite* (also *Dedekind finite*) if there is no one-to-one map from X into a proper subset of X.
- *V-finite* if $X = \emptyset$ or there is no one-to-one map from $2 \times X$ into X.
- *VI-finite* if X is empty, if it is a singleton, or if there is no one-to-one map from $X \times X$ into X.
- *VII-finite* if X is I-finite or it is not well-orderable.

Remarks 1. 1. To each notion of finiteness $\mathcal{Q}$ corresponds a notion of infinity: we will say that a set is *$\mathcal{Q}$-infinite* if it is not $\mathcal{Q}$-finite.

2. A set X is I-finite if and only if there is a bijection between X and an ordinal $n < \omega$. This one is the commonly accepted definition of finiteness, among other reasons, because it is absolute for models of ZF. Therefore, whenever we use the terms *finite* and *infinite* without further qualification, we mean I-finite and I-infinite, respectively.
3. An infinite set X which is Ia-finite is also called *amorphous*. An Ia-infinite set is also called *partible*.
4. A set X is IV-infinite if and only if it contains a well-orderable infinite subset. A set X is III-infinite if and only if $\mathcal{P}(X)$ is IV-infinite.
5. Defining cardinal numbers, as well as operations and order on cardinal numbers, without AC (see Appendix A; also, Jech [Jec73]), we can rewrite some of the definitions in the following way:
    - X is I-finite iff $|X| < \aleph_0$.
    - X is III-finite iff $2^{|X|} + 1 > 2^{|X|}$.
    - X is IV-finite iff $|X| + 1 > |X|$ iff $|X| \not\geq \aleph_0$.
    - X is V-finite iff $|X| = 0$ or $2 \cdot |X| > |X|$.

- X is VI-finite iff $|X| = 0, 1$ or $|X|^2 > |X|$.

The proof of the following theorem can be found in Lévy [Lév58]:

Theorem 2 (Lévy). *If X is II-finite and linearly orderable, then X is I-finite.* □

Given two notions of infinity $\mathcal{Q}$ and $\mathcal{Q}'$, we use the abbreviation $\mathcal{Q} \longrightarrow \mathcal{Q}'$ instead of the formula $\forall X(X$ is $\mathcal{Q}$-finite $\rightarrow X$ is $\mathcal{Q}'$-finite). It is shown in Lévy [Lév58] that

$$\mathrm{I} \longrightarrow \mathrm{Ia} \longrightarrow \mathrm{II} \longrightarrow \mathrm{III} \longrightarrow \mathrm{IV} \longrightarrow \mathrm{V} \longrightarrow \mathrm{VI} \longrightarrow \mathrm{VII},$$

and it is also shown that none of these implications can be reversed in ZFA. For ZF, the same independence results were established by Jech and Sochor [Jec66]; also, Spišiak and Vojtáš [Spi88] determined what combinations of sets which are finite according to any of the definitions above can coexist in a model of ZF.

Spišiak and Vojtáš [Spi88] introduce the following way to derive new definitions of finiteness from old ones.

Definition 3. Let $\mathcal{Q}$ be any notion of finiteness. We say that a set X is *$\mathcal{Q}''$-finite* if $\mathcal{P}(X)$ is $\mathcal{Q}$-finite.

Remarks 2. 1. If $\mathcal{Q}_1, \mathcal{Q}_2$ are notions of finiteness and $\mathcal{Q}_1 \longrightarrow \mathcal{Q}_2$, then $\mathcal{Q}_1'' \longrightarrow \mathcal{Q}_2''$.

2. IV$''$ is equivalent to III.

3. III$''$ is equivalent to I, by a result of Tarski [Tar24] which states that for each I-infinite set X, $\mathcal{P}(\mathcal{P}(X))$ is IV-infinite. As a consequence, I$''$, Ia$''$, and II$''$ are also equivalent to I.

4. We have that VII $\longrightarrow$ VII$''$: If X is not well orderable, then $\mathcal{P}(X)$ is not well orderable, since X can be mapped one-to-one into $\mathcal{P}(X)$.

5. VII$''$ does not satisfy (1) of Degen's criterion for a definition of finiteness. In fact, in any model of ZF where $\mathcal{P}(\omega)$ is not well orderable we have that *all* sets are VII$''$-finite: If X is not well orderable, then it is VII$''$-finite (by the previous remark). If X is I-infinite and well orderable, then $\mathcal{P}(\omega)$ can be mapped one-to-one into $\mathcal{P}(X)$, and consequently $\mathcal{P}(X)$ cannot be well ordered. Of course, this

makes VII″ a rather unreasonable definition for finiteness, but it still satisfies the formal requirement of being equivalent to I under AC.

Spišiak and Vojtáš [Spi88] provide a few more relations that can be easily proved using elementary cardinal arithmetic.

Theorem 4 (Spišiak and Vojtáš). *The following implications are provable from ZF or ZFA:*

1. *III $\longrightarrow$ V″*
2. *V″ $\longrightarrow$ IV*
3. *VI″ $\longrightarrow$ V.*

Proof. Let X be a set and let $\mathfrak{a}$ be its cardinal.

For 1, assume that X is III-finite. Then, by Remark 2.2, X is IV″-finite, and by Remark 2.1, X is V″-finite.

For 2, assume that X is IV-infinite; we will show that X is also V″-infinite. We have that $\mathfrak{a} + 1 = \mathfrak{a}$, and therefore $2^{\mathfrak{a}+1} = 2^{\mathfrak{a}}$. This means that $2^{\mathfrak{a}}$ is idemmultiple, that is, that $2 \cdot 2^{\mathfrak{a}} = 2^{\mathfrak{a}}$. Therefore, $\mathcal{P}(X)$ is V-infinite and X is V″-infinite.

Finally, for 3, assume that X is V-infinite. That means that $2 \cdot \mathfrak{a} = \mathfrak{a}$; consequently, $2^{2 \cdot \mathfrak{a}} = 2^{\mathfrak{a}}$, that is, $(2^{\mathfrak{a}})^2 = 2^{\mathfrak{a}}$. In other words, $\mathcal{P}(X)$ is idempotent, that is, VI-infinite, and therefore X is VI″-infinite. □

These derived notions are further studied by Spišiak [Spi93] and Howard and Spišiak [How94].

Another group of notions of finiteness was introduced by Truss [Tru74] in the form of classes of Dedekind finite cardinals.

Definition 5. Define the following classes of cardinals:

$\Delta_1 = \{\mathfrak{a} : \mathfrak{a} = \mathfrak{b} + \mathfrak{c} \rightarrow \mathfrak{b}$ or $\mathfrak{c}$ is finite$\}$

$\Delta_2 = \{|X| :$ any linearly ordered partition of X is I-finite$\}$

$\Delta_3 = \{|X| :$ any linearly ordered subset of X is I-finite$\}$

$\Delta_4 = \{\mathfrak{a} : \neg(\aleph_0 \leq^* \mathfrak{a})\}$ (see Appendix A for the definition of $\leq^*$)

$\Delta_5 = \{\mathfrak{a} : \neg(\mathfrak{a} + 1 \leq^* \mathfrak{a})\}$.

Remarks 3. 1. We will use the symbols Δ_i, for $i = 1, \ldots, 5$, as notions of finiteness; that is, we will say that a set X is Δ_i-finite if $|X| \in \Delta_i$.

2. Clearly, Δ_1-finiteness is equivalent to Ia-finiteness. It can also be proved that Δ_2-finiteness is equivalent to II-finiteness and that Δ_4-finiteness is equivalent to III-finiteness.

Truss [Tru74] proves that the following implications can be proved from ZF or ZFA:

$$\mathrm{I} \longrightarrow \Delta_1 \longrightarrow \Delta_2 \longrightarrow \Delta_4 \longrightarrow \Delta_5 \longrightarrow \mathrm{IV}$$

and

$$\Delta_2 \longrightarrow \Delta_3 \longrightarrow \mathrm{IV}.$$

Also, he proves that none of these implications is reversible.

Howard and Yorke [How89] introduce yet another notion:

Definition 6. A set X is *D-finite* if either it has at most one element, or $X = X_1 \cup X_2$, with $|X_1|, |X_2| < |X|$.

A D-finite set with more than one element is called *decomposable.* They prove that IV $\longrightarrow D \longrightarrow$ VII, and that $D \not\longrightarrow$ VI. It is unclear whether V $\longrightarrow D$ or not.

In the same paper the authors introduce the following principles: If $\mathcal{Q}_1, \mathcal{Q}_2$ are notions of finiteness, $E(\mathcal{Q}_1, \mathcal{Q}_2)$ stands for the formula $\forall X(X$ is $\mathcal{Q}_1$-finite $\leftrightarrow$ X is $\mathcal{Q}_2$-finite$)$. Several of these principles are (or are equivalent to) well known weak principles of choice (see Howard and Rubin [How98], Note 94). In particular, $E(\mathrm{V}, \mathrm{VI})$, $E(\mathrm{VI}, \mathrm{VII})$, and $E(\mathrm{I}, D)$ are equivalent to AC.

We summarize the relationships between the various notions of finiteness in Figure 1.1. In this diagram none of the arrows can be reversed, and, in addition, we have VI$''$ $\not\longrightarrow$ IV, IV $\not\longrightarrow$ VI$''$, and $D \not\longrightarrow$ VI. The relationship between Δ_5 and V$''$ as well as the relationship between D and V have not been established, while the relationship between Δ_3 and III is solved by the following theorem:

Theorem 7. *It is consistent with ZF to have sets X, Y such that X is III-finite but Δ_3-infinite while Y is Δ_3-finite but III-infinite.*

The proof can be found in Section 1.5.

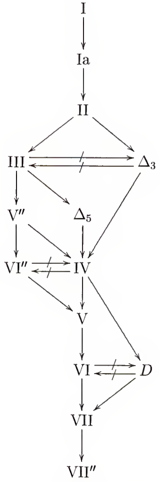

Figure 1.1: Relations Between Notions of Finiteness

1.3 Axioms of Choice for Finite Families

In this section we introduce two kinds of principles of choice, stated as axioms of choice for families that are finite according to one of the notions introduced in the previous section.

Given a family X of non-empty sets (that is, neither $\emptyset$ nor any atom is a member of X), a *choice function* is a function $f : X \to \bigcup X$ such that

$$\forall x \in X(f(x) \in x).$$

Definition 8. Let $\mathcal{Q}$ be a notion of finiteness. The principle $C(\mathcal{Q})$ is the sentence:

$$\forall X(X \text{ is a } \mathcal{Q}\text{-finite family of non-empty sets} \to X \text{ has a choice function})$$

The principle $C^{-}(\mathcal{Q})$ is the sentence:

$$\forall X(X \text{ is a } \mathcal{Q}\text{-finite but I-infinite family of non-empty sets} \to$$
$$\exists Y \subset X(Y \text{ is I-infinite and has a choice function})).$$

If f is a choice function for an infinite family Y contained in a set X, we say that f is a *partial choice function* for X.

Remarks 4. The following are all theorems of ZF and of ZFA:

1. $C(\mathrm{I})$ is true. Also, $C^{-}(\mathrm{I})$ is vacuously true in both theories.
2. If $\mathcal{Q}$ is a notion of finiteness, then $C(\mathcal{Q})$ implies $C^{-}(\mathcal{Q})$.
3. If $\mathcal{Q}_1, \mathcal{Q}_2$ are notions of finiteness and $\mathcal{Q}_1 \longrightarrow \mathcal{Q}_2$, then $C(\mathcal{Q}_2)$ implies $C(\mathcal{Q}_1)$ and $C^{-}(\mathcal{Q}_2)$ implies $C^{-}(\mathcal{Q}_1)$.
4. The principle $E(\mathrm{I}, \mathcal{Q})$ implies $C(\mathcal{Q})$, for each notion of finiteness $\mathcal{Q}$ in Figure 1.1.

Lemma 9. *If $C(\mathcal{Q})$ holds and $\mathcal{P}(X) \setminus \{\emptyset\}$ is $\mathcal{Q}$-finite, then X is well orderable.*

Proof. Once we have a choice function for $\mathcal{P}(X) \setminus \{\emptyset\}$, it is enough to repeat the usual proof of the fact that AC implies the Well Ordering Principle. □

Corollary 10. *C(VII) is equivalent to AC.*

Proof. Clearly, AC implies $C(\mathrm{VII})$. Suppose X is a non-well-orderable set. Then $\mathcal{P}(X)$ is also non-well-orderable, and by Lemma 9 we conclude that X is actually well orderable. Contradiction. □

Corollary 11. *If $\mathcal{Q}$ is one of the notions of finiteness in Figure 1.1, and $C(\mathcal{Q})$ holds, then we have $E(I, \mathcal{Q}'')$.*

Proof. Suppose $C(\mathcal{Q})$ holds and that X is $\mathcal{Q}''$-finite but I-infinite. Since $\mathcal{P}(X)$ is $\mathcal{Q}$-finite, Lemma 9 guarantees that X is well orderable.

Being I-infinite and well orderable, X cannot be VII-finite, and therefore, it is not finite according to any of the other definitions in Figure 1.1, except VII″. Therefore, the only case when this can happen is when $\mathcal{Q}$ is VII. However, the assumption $C(\text{VII})$ is equivalent to AC, and this implies $E(\text{I}, \text{VII}'')$. □

Next we prove another easy result:

Lemma 12. *$C(Ia)$ and $C^{-}(Ia)$ are equivalent.*

Proof. We know that $C(\text{Ia})$ implies $C^{-}(\text{Ia})$. To prove the converse implication, assume that X is an infinite Ia-finite family of non-empty sets and that $C^{-}(\text{Ia})$ holds. Let f be a partial choice function for X, with domain $X' \subset X$; since X' is I-infinite, $X \setminus X'$ is I-finite, and therefore it has a choice function g. Then $f \cup g$ is a choice function for X. □

The preceding result can be proved from both ZF and ZFA. However, it is vacuously true in ZF, as the next two results show. We will see in Section 1.4 that it is not vacuously true in ZFA, since we will show that Theorem 13 and Corollary 14 cannot be proved in ZFA.

Theorem 13 (ZF). *Let X be an infinite set. If every family of non-empty sets indexed by X has a partial choice function, then X is partible (that is, Ia-infinite).*

Proof. For the sake of a contradiction, suppose that X is Ia-finite. Since every subset of X is either I-finite or the complement of a I-finite set, when a formula holds for infinitely many elements of X, we will say that the formula is true "almost everywhere."

Given any function f with infinite domain contained in X, we define $\rho(f)$ as the unique ordinal α such that $\{x \in X : \mathrm{rank}(f(x)) = \alpha\}$ is infinite (such α is unique because any well ordered partition of X has exactly one infinite part). If f, g are functions such that $\mathrm{dom}(g) \subset \mathrm{dom}(f) \subset X$, and $\mathrm{dom}(g)$ is infinite, we will write $g \prec f$ if for all $x \in \mathrm{dom}(g)$, $g(x) \in f(x)$. Clearly, if $g \prec f$, then $\rho(g) < \rho(f)$.

Choose an f_0 with minimum ρ such that $\mathrm{range}(f_0)$ is infinite (there is at least one such function with infinite range, namely, the identity function on X); we can assume that $\emptyset \notin \mathrm{range}(f_0)$. Since we can see f_0 as a family of sets indexed by X (by taking any undefined values of f_0 to be an arbitrary non-empty set), by hypothesis there exists a partial choice function $g_0 : Y \subset \mathrm{range}(f_0) \to \bigcup(\mathrm{range}(f_0))$. Then $h_0 = g_0 \circ f_0$ has an infinite domain contained in X, and we have that $h_0 \prec f_0$. Therefore, $\rho(h_0) < \rho(f_0)$, and consequently h_0 has a finite range. Since X is Ia-finite, that means that h_0 is constant almost everywhere.

Let $K = \{c : \exists h \prec f_0, h \equiv c \text{ almost everywhere}\}$, and define $f_1(x) = f_0(x) \triangle K$ for all $x \in \mathrm{dom}(f_0)$. If $f_1 \equiv \emptyset$ almost everywhere, then $f_0 \equiv K$ almost everywhere, contradicting the fact that $\mathrm{range}(f_0)$ is infinite. Therefore, f_1 is a family of non-empty sets defined almost everywhere on X. Using the hypothesis we can find $h_1 \prec f_1$ just as we found h_0 above.

Now, for all $c \in K$ we have that $\mathrm{rank}(c) < \rho(f_0)$. Therefore $\rho(h_1) < \rho(f_0)$, since the images of h_1 are either in K or in $f_0(x)$ for some $x \in X$. Thus, h_1 has finite range, and consequently it is constant almost everywhere with value, say, c_1.

We arrive to a contradiction as follows: $c_1 \in K$ if and only if $c_1 \in f_0(x)$ for almost all $x \in X$. Therefore, $c_1 \notin f_0(x) \triangle K = f_1(x)$ for almost all $x \in X$. This is not possible, since $h_1 \prec f_1$. □

Corollary 14 (ZF). *$C^-(Ia)$ is equivalent to $E(I, Ia)$.*

Proof. As we have already seen, if $E(\mathrm{I}, \mathrm{Ia})$ holds, then $C^-(\mathrm{Ia})$ is vacuously true.

Suppose that $C^{-}(\mathrm{Ia})$ holds and that X is a I-infinite Ia-finite set. By Theorem 13, there exists a family $f : X \to V$ of non-empty sets indexed by X with no partial choice function.

If f is constant almost everywhere with value, say, c, and $d \in c$, then the function g, constantly equal to d and with domain equal to $f^{-1}(c)$, would be a partial choice function for f. Thus, f cannot be constant almost everywhere, and therefore it must have an infinite range. But then range(f) is a Ia-finite set, since any partition of range(f) in two I-infinite sets induces a partition of X in two I-infinite sets. The hypothesis of this corollary implies that the set range(f) has a partial choice function, which means that the family f has a partial choice function. Contradiction. □

The next result shows that it is impossible to have at the same time I-infinite, IV-finite families of sets, and total choice for all of them. But even more, since the result is a theorem of ZFA as well as of ZF, starting with a I-infinite, IV-finite set of atoms leads to exactly the same contradiction.

Theorem 15. *$C(IV)$ implies $E(I, IV)$.*

Proof. Suppose that $C(\mathrm{IV})$ holds, and that X is a I-infinite, IV-finite set. Consider the set $X_*^{<\infty}$ of all finite sequences from X that do not repeat elements.

We claim that $X_*^{<\infty}$ is also IV-finite.[1] Suppose it is not; then there exists a countable subset $\{s_n : n < \omega\} \subset X_*^{<\infty}$. Clearly, $X' = \bigcup_{n<\omega} \mathrm{range}(s_n)$ is infinite, because only finitely many sequences without repetitions can be taken from a finite set. Now we define a well order on X': If a is the i-th element of s_n and b is the j-th element of s_m, then $a \prec b$ if and only if

1. $n < m$, and b does not appear in s_k for any $k \leq n$, or
2. $n = m$, $i < j$, and b does not appear in s_k for any $k < n$.

[1] This claim appears in Jech [Jec78] as an exercise.

This way, X' is an infinite well ordered subset of X, contradicting the assumption that X is IV-finite.

Now, since $X_*^{<\infty}$ is IV-finite, so is the family

$$F = \{X_s = (X \setminus \operatorname{range}(s)) \times \{s\} : s \in X_*^{<\infty}\},$$

because the map $s \mapsto X_s$ is one-to-one. Therefore, by hypothesis, F has a choice function f.

We find a contradiction by defining a non-repeating ω-sequence from elements of X, thereby showing that it cannot be IV-finite. Choose an initial element $x_0 \in X$; the rest of the sequence is defined by recursion. Assuming that $\langle x_0, \ldots, x_k\rangle$ has already been defined, we define x_{k+1} by

$$x_{k+1} = \left(f(X_{\langle x_0,\ldots,x_k\rangle})\right)_1.$$

$\square$

The same result can be obtained for Δ_3, using a similar proof. However, the first part of the proof is slightly more laborious in this case.

Theorem 16. *$C(\Delta_3)$ implies $E(I, \Delta_3)$.*

Proof. Assume that $C(\Delta_3)$ holds, and suppose that X is a I-infinite, Δ_3-finite set.

We claim that $X_*^{<\infty}$, defined as before, is Δ_3-finite. Suppose that it is not, and let S be an infinite subset of $X_*^{<\infty}$ which has a linear order, say, $\prec$.

Case 1. There exists $n \in \omega$ such that $S_n = \{s \in S : \operatorname{length}(s) = n\}$ is I-infinite. Then, let s_0 be a sequence of maximal length such that $\{s : s_0{}^\frown s \in S_n\}$ is infinite (notice that $\operatorname{length}(s_0) < n$). Then the set $X' = \{x \in X : s_0{}^\frown\langle x\rangle^\frown s \in S_n,$ for some $s \in X_*^{<\infty}\}$ must be infinite, and for each $x \in X'$, the set of completions $\{s \in X_*^{<\infty} : s_0{}^\frown\langle x\rangle^\frown s \in S_n\}$ must be finite. We can define now a one-to-one mapping f from X' into S as follows: for each $x \in X'$, let $f(x)$ be the $\prec$-smallest sequence of the form $s_0{}^\frown\langle x\rangle^\frown s$.

This map induces a linear order on the infinite subset X' of X, contradicting the assumption that X is Δ_3-finite.

Case 2. For all $n \in \omega$, $S_n = \{s \in S : \text{length}(s) = n\}$ is I-finite. In this case, we can choose s_n to be the $\prec$-smallest element from S_n, for each n such that $S_n \neq \emptyset$. There have to be infinitely many values of n for which $S_n \neq \emptyset$, because S could not be infinite otherwise. Therefore, $\{s_n : S_n \neq \emptyset\}$ is an infinite subset of S, and it is well ordered. As we saw in the first half of the proof of Theorem 15, this means that X is IV-infinite, contradicting the assumption that X is Δ_3-finite.

Now, since $X_*^{<\infty}$ is Δ_3-finite, the family

$$F = \{X_s = (X \setminus \text{range}(s)) \times \{s\} : s \in X_*^{<\infty}\}$$

is also Δ_3-finite, so by hypothesis it has a choice function f. We can then repeat the last argument of the proof of Theorem 15 to conclude that X has a countably infinite subset and reach a contradiction with the assumption that X is Δ_3-finite. □

Some of the implications found in this section are summarized in Figure 1.3

1.4 Independence Proofs: Permutation Models

The technique of using what are now called *permutation models* to obtain independence results, especially for AC and related principles, was introduced originally by Fraenkel in 1922, and later developed by Mostowski. The main limitation of the technique is that it provides independence results only for theories weaker than ZF, like ZF minus the axiom of foundation or ZFA. However, it still provides useful independence results in ZF when combined with the transfer theorems of Jech-Sochor and Pincus.

To create a permutation model we start with a model $\mathcal{M}$ of ZFA + AC+*the set A of atoms is I-infinite*; the consistency of this theory can be proved from the assumption that ZF is consistent. Such a model has many $\in$-automorphisms, one for

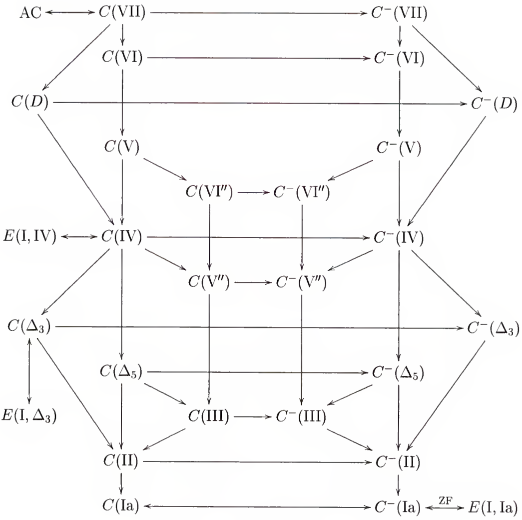

Figure 1.2: Some implications between principles

each permutation of the set A. Roughly speaking, the permutation model will be the class containing all the objects (sets or atoms) which have a "large" subgroup of symmetries (automorphisms that leave the object invariant), inside of a prescribed basic group $\mathcal{G}$ of symmetries.

The Definition of Permutation Models

Given a model $\mathcal{M}$ of ZFA+AC+*the set A of atoms is I-infinite*, and a permutation π of the A, we can extend π to an $\in$-automorphism of $\mathcal{M}$, by recursion on the well founded relation $\in$, in the following way:

1. If $a \in A$, the extension agrees with the permutation.

2. $\pi\emptyset = \emptyset$.

3. If πy is defined for all $y \in x$, we define $\pi x = \pi``x$.

Given a group $\mathcal{G}$ of permutations of A and an object (atom or set) x in $\mathcal{M}$, we define the following subgroup of $\mathcal{G}$:

$$\mathrm{sym}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} : \pi x = x\},$$

and if x is a set, we define:

$$\mathrm{fix}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} : \pi y = y, \text{ for all } y \in x\}.$$

When it is clear from the context what group $\mathcal{G}$ are we using, we will drop the subscript.

A *normal filter* $\mathcal{F}$ on $\mathcal{G}$ is a filter in the lattice of subgroups of $\mathcal{G}$ which is closed under conjugation; that is, for all $\pi \in \mathcal{G}$, for all $H \in \mathcal{F}$, $\pi H \pi^{-1} \in \mathcal{F}$.

A *permutation submodel* $\mathcal{N}$ of $\mathcal{M}$ (or simply a *permutation model*) is determined by a group $\mathcal{G}$ of permutations of A and a normal filter $\mathcal{F}$ on $\mathcal{G}$. If $\mathcal{G}, \mathcal{F}$ are fixed, an object $x \in \mathcal{M}$ is said to be *symmetric* if $\mathrm{sym}(x) \in \mathcal{F}$. Then the class $\mathcal{N}$ is formed by all the hereditarily symmetric objects; that is,

$$\mathcal{N} = \{x : \mathrm{sym}(x) \in \mathcal{F} \text{ and for all } y \in x,\, y \in \mathcal{N}\}.$$

Clearly, $A \in \mathcal{N}$, and every pure set (that is, a set with no atoms in its transitive closure) is also in $\mathcal{N}$. The proof of the following fundamental theorem can be found in Jech [Jec73].

Theorem 17. *$\mathcal{N}$, defined as above, is a model of ZFA.* □.

In most cases, the filter $\mathcal{F}$ is obtained from a *normal ideal* on A, which is an ideal of subsets of A which contains all the singletons and is closed under images by elements of $\mathcal{G}$. If I is such an ideal, then we obtain a normal filter $\mathcal{F}$ on $\mathcal{G}$ by

$$\mathcal{F} = \{H < \mathcal{G} : H \supset \mathrm{fix}(E), \text{ for some } E \in I\}.$$

It is easy to see that $\mathcal{F}$ is indeed a normal filter. In this case, x is symmetric with respect to $\mathcal{F}$ if and only if there exists $E \in I$ such that $\text{fix}(E) \subset \text{sym}(x)$; such an E is called a *support* for x. We refer to I as the *ideal of supports* for $\mathcal{N}$.

We will now describe five permutation models, and describe some of their properties. Although these are models of ZFA, we will use them later to establish some results in ZF.

The Basic Fraenkel Model (BFM)

The first permutation model we introduce is the one usually called *Basic Fraenkel Model.* It is constructed from a model of ZFA + AC where the set A of atoms is countably infinite, by taking the group $\mathcal{G}$ of all permutations of A, and using the ideal of finite subsets of A as the ideal of supports. In BFM, A is an amorphous set.

It is established by Spišiak and Vojtáš [Spi88] and Spišiak [Spi93] that for most pairs $\mathcal{Q}$, $\mathcal{Q}'$ of different notions of finiteness from diagram 1.1, there is a set in BFM that is finite according to one of the two notions, but not according to the other. The only possible exception is the pair II, III; the question of whether these two notions are equivalent in BFM is listed as open in Spišiak [Spi93]. In this section we will establish two results about principles from Section 1.3 which hold in BFM. First, we need some lemmas.

Lemma 18. *The model BFM has the following property: If E supports x, and π, π' are permutations of A such that $\pi \restriction E = \pi' \restriction E$, then $\pi x = \pi' x$.*

Proof. It is clear that $\pi^{-1} \circ \pi' \in \text{fix}(E)$, so $\pi^{-1} \circ \pi' x = x$. That means that $\pi' x = \pi x$. □

Lemma 19. *The model BFM has the following property: Suppose that π is a permutation of A such that $\pi x = x$ for some object x, and suppose that a is an atom such that there exists a support E' of x that does not contain either πa or $\pi^{-1} a$. Then there exists a support $E \subset E'$ of x that does not contain a.*

Proof. If $\pi a = a$, the result is clear. Assume that $\pi a = b \neq a$, and suppose that every support of x contains a. Let $E \cup \{a\}$ be a support for x such that $b, \pi^{-1}a \notin E$; we will show that E is also a support for x.

Let $\sigma \in \mathrm{fix}(E)$, and let $c = \sigma a$. We want to show that $\sigma x = x$, and by Lemma 18, it is enough to show that the transposition (a, c) satisfies $(a, c)x = x$, since $(a, c) \restriction (E \cup \{a\}) = \sigma \restriction (E \cup \{a\})$.

If $c = a$, then $\sigma x = x$. Suppose then that $c \notin E$, so the transposition (b, c) is in $\mathrm{fix}(E)$. Define the permutation π' as follows: for all $d \in A$,

$$\pi' d = \begin{cases} b, & \text{if } d = a; \\ a, & \text{if } d = b; \\ \pi d, & \text{if } d \in E \text{ and } \pi d \in E; \\ e, & \text{if } d = \pi^k e,\ e \in E,\ d \notin E,\ e \neq \pi e' \text{ for any } e' \in E, \\ & \quad \text{and } \pi^i e \in E \text{ for all } i < k; \\ d, & \text{otherwise.} \end{cases}$$

Then π' satisfies $\pi' \restriction (E \cup \{a\}) = \pi \restriction (E \cup \{a\})$, and therefore, by Lemma 18, $\pi' x = \pi x = x$. We have then that

$$(\pi')^{-1} \circ (b, c) \circ \pi' x = x,$$

and it is easy to check that $(\pi')^{-1} \circ (b, c) \circ \pi' = (a, c)$, *as long as* $c \notin (\pi')``E$.

If $c \in (\pi')``E$, we choose $c'$ in $A \setminus (E \cup \{a\} \cup (\pi')``E)$. Just as before, $(\pi')^{-1} \circ (b, c') \circ \pi' x = x$, and $(\pi')^{-1} \circ (b, c) \circ \pi' = (a, c')$. Since $(c, c') \in \mathrm{fix}(E \cup \{a\})$, we have that

$$(a, c)x = (c, c')(a, c')(c, c')x = x.$$

This finishes the proof. $\square$

Theorem 20. *C^-(VII) holds in BFM.*

Proof. Let X be a non-well-orderable (that is, VII-finite but I-infinite) family of non-empty sets; we want to prove that there exists a I-infinite subset $X' \subset X$ with a choice function.

Let E be a support for X. If every element $x \in X$ were also supported by E, E would be also a support for a bijection between X and an ordinal, which is not possible. Take then x_0 such that it is not supported by E, and let E_0 be a support for x_0 of minimum size such that $E \subset E_0$.

Since $x_0 \neq \emptyset$, we can choose an element $y_0 \in x_0$, and let E_1 be a support for y_0 such that $E_0 \subset E_1$. Pick $a_0 \in E_0 \setminus E$, and call $E_2 = E_1 \setminus \{a_0\}$. Define

$$f = \{\pi(\langle x_0, y_0\rangle) : \pi \in \text{fix}(E_2)\}.$$

Claim. f is a function. Indeed, if $\pi x_0 = \pi' x_0$ but $\pi y_0 \neq \pi' y_0$, that means that $\pi a_0 \neq \pi' a_0$, and then $\pi^{-1} \circ \pi' x_0 = x_0$, but $\pi^{-1} \circ \pi' a_0 \neq a_0$. Since $\pi^{-1} \circ \pi' \in \text{fix}(E_0 \setminus \{a\})$, we have that neither $\pi^{-1} \circ \pi' a_0$ nor $(\pi^{-1} \circ \pi')^{-1} a_0$ are members of $E_0 \setminus \{a_0\}$. Therefore, we can apply Lemma 1.4 to conclude that there is a support for x_0 contained in $E_0 \setminus \{a_0\}$; this contradicts the choice of E_0 as a support for x_0 of minimum size.

Claim. $\text{dom}(f) \subset X$, and it is I-infinite. In fact, for all $\pi \in \text{fix}(E_2)$, we have that $\pi x_0 \in \pi X = X$. Also, for every $b, b' \notin E_2$, $b \neq b'$, the elements $(a_0, b)x_0$ and $(a_0, b')x_0$ must be different: Suppose they are not, and take $\pi = (a_0, b')(a_0, b)$. We have that $\pi x_0 = x_0$, while $\pi a_0 = b$ and $\pi^{-1} a_0 = b'$; then, using Lemma 1.4, we can find a support for x_0 contained in $E_0 \setminus \{a_0\}$, and again this is a contradiction.

Finally,

Claim. f is a choice function for its domain. Clearly, for all $\pi \in \text{fix}(E_2)$, $\pi y_0 \in \pi x_0$.

□

This result shows that Corollary 14 cannot be proved in ZFA. Also, it is easy to check that Theorem 13 fails in BFM, and therefore is not provable in ZFA: We have

that every family of non-empty sets $\{x_a : a \in A\}$ indexed by A is either I-finite or amorphous; in the second case, by Theorem 20, $\{x_a : a \in A\}$ has a partial choice function. However, A is amorphous in BFM.

Another consequence of Theorem 20 is that $C^-(\mathrm{VII})$ does not imply $C(\mathrm{IV})$ in ZFA, since $C(\mathrm{IV})$ does not hold in BFM: if it did hold, then, by Theorem 15, we would have $E(\mathrm{I}, \mathrm{IV})$ in BFM, which is not true because A is I-infinite Ia-finite set. Similarly, using Theorem 16, we conclude that $C^-(\mathrm{VII})$ does not imply $C(\Delta_3)$ in ZFA. It is not known whether $C^-(\mathrm{VII})$ implies $C(\mathrm{III})$ or $C(\mathrm{II})$ (in ZFA or ZF).

We will briefly describe now the last four permutation models. All the properties listed here are well known, and they establish the non-equivalence of several notions of finiteness. However, the main reason for including these models here is that they will be used in Section 1.6 to obtain independence proofs for several principles of the form $C(\mathcal{Q})$ and $C^-(\mathcal{Q})$.

The Second Fraenkel Model (SFM)

To construct the Second Fraenkel Model we start with a model of ZFA with a countable set of atoms. In order to describe the group $\mathcal{G}$, we organize the set of atoms in countably many pairs:

$$A = \bigcup_{n \in \omega} \{a_n, b_n\}.$$

We take $\mathcal{G}$ to be the group of all permutations of A that leave invariant each pair $\{a_n, b_n\}$, for $n \in \omega$, and we take the ideal of finite subsets of A as our normal ideal I.

The following two results can be found in Truss [Tru74]:

Lemma 21 (Truss). *In SFM, A is an infinite Δ_5-finite set.* □

Lemma 22 (Truss). *SFM satisfies $E(I, III)$.* □

Consequently, it cannot be proved in ZFA that every Δ_5-finite set is III-finite.

A Modification of the Basic Fraenkel Model

We call BFM($\aleph_1$) a model constructed in a similar way to BFM, but where $|A| = \aleph_1$ in $\mathcal{M}$. Again, we take the group $\mathcal{G}$ to be the group of all the permutations of A, but in this case we take I to be the ideal of the (at most) *countable* subsets of A.

The following results are well known in the literature. They can be found, for example, in [Spi93]:

Lemma 23. *In BFM($\aleph_1$), A is an infinite V-finite set.* □

Lemma 24. *The Countable Axiom of Choice holds in BFM($\aleph_1$).* □

Corollary 25. *The principle $E(I, IV)$ holds in BFM($\aleph_1$).* □

Therefore, it cannot be proved in ZFA that every V-finite set is IV-finite.

Mostowski's Ordered Model (MOM)

To construct this model, we start with a model of ZFA with a countable set A atoms, ordered in the order type of the set of rational numbers. We choose $\mathcal{G}$ to be the group of all order preserving permutations of A, and I to be the ideal of finite subsets of A.

Theorem 26 (Mostowski). *In MOM, every set can be linearly ordered.* □

A proof of the previous theorem can be found in Jech [Jec73]. Now, by Theorem 2, we obtain the following lemma:

Lemma 27. *The principle $E(I, II)$ holds in MOM.* □

In Lévy [Lév58], the following is also proved:

Lemma 28. *The set A is III-finite in MOM.* □

Consequently, it cannot be proved in ZFA that every III-finite set is II-finite.

The Mathias-Pincus Model I (MPM-I)

To construct our last permutation model, we start with a model of ZFA where A is countable and there is a partial order $\preceq$ on A which is a *universal homogeneous partial order*, that is, the following two conditions hold:

1. Every finite partially ordered set can be embedded in $\langle A, \preceq \rangle$.
2. For every two finite sets $E, E' \subset A$ such that there is an isomorphism ϕ between $\langle E, \preceq \rangle$ and $\langle E', \preceq \rangle$, there exists an automorphism of $\langle A, \preceq \rangle$ that extends ϕ.

A proof of the existence of a countable homogeneous partial order can be found in Jech [Jec73], although it also follows from more general results.

The group $\mathcal{G}$ is taken to be the group of all automorphisms of the structure $\langle A, \preceq \rangle$, and we take I to be the ideal of finite subsets of A. The following two properties are listed in Howard and Rubin [How98]:

Lemma 29. *The set A is an infinite II-finite set.* □

Lemma 30. *There are no amorphous sets in MPM-I; that is, MPM-I satisfies the principle $E(I, Ia)$.* □

This way, it cannot be proved in ZFA that every II-finite set is Ia-finite.

1.5 A ZF Model

In this section we use Cohen's original model to obtain some independence results.

First, we will quickly review the basic concepts of symmetric submodels of generic extensions, also called *symmetric extensions* or *symmetric models*, since this tool will be used both in this section and the next one.

It seems plausible that every consistency result obtained using symmetric extensions can also be obtained using models of the form $\mathrm{HOD}^{\mathcal{M}[G]}(A)$, (the class of *hereditarily ordinal definable* sets in $\mathcal{M}[G]$ with parameters from A), where $\mathcal{M}[G]$ is a generic extension of the ground model $\mathcal{M}$ and A is a set or a class in $\mathcal{M}[G]$; indeed, we will use this kind of model in Chapter 2. The advantage of using HOD models is that one can explicitly decide which objects from $\mathcal{M}[G]$ must be in the submodel, while symmetric extensions must be designed in such a way that the desired objects remain in the submodel; in other words, HOD models are constructed by adding the desired objects, while symmetric models are designed by subtracting the undesired objects and then proving that the desired objects are still there. However, there is

much more work done using symmetric extensions, especially when dealing with weak principles of choice, and there is no point in rewriting much of the theory, when many results can be quoted from well known sources like Jech [Jec73].

The Definition of Symmetric Models

Let $\mathcal{M}$ be a model of ZFC, let B be a complete Boolean algebra in $\mathcal{M}$, and let $\mathcal{M}^B$ be the corresponding Boolean-valued model, that is, the class of names. If π is an automorphism of B, we can extend π to $\mathcal{M}^B$ as follows, by recursion on the well founded relation $x \prec y$ iff $x \in \mathrm{dom}(y)$:

1. $\pi\emptyset = \emptyset$.
2. If $\pi(x)$ has been defined for all $x \in \mathrm{dom}(y)$, we define $\mathrm{dom}(\pi y) = \pi``\mathrm{dom}(y)$, and $(\pi y)(\pi x) = \pi(y(x))$ for all $\pi x \in \mathrm{dom}(\pi y)$.

This extension, which we have called also π, is a one-to-one function from $\mathcal{M}^B$ into $\mathcal{M}^B$. If $\check{x}$ is the canonical name for a set $x \in \mathcal{M}$, then $\pi\check{x} = \check{x}$, for all automorphisms π of B.

Let $\mathcal{G}$ be a group of automorphisms of B. For each $x \in \mathcal{M}^B$, let

$$\mathrm{sym}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} : \pi(x) = x\}.$$

Clearly, $\mathrm{sym}_{\mathcal{G}}(x)$ is a subgroup of $\mathcal{G}$. A set $\mathcal{F}$ of subgroups of $\mathcal{G}$ is called a *normal filter* on $\mathcal{G}$ if $\mathcal{F}$ is a filter in the lattice of subgroups of $\mathcal{G}$, and for all $H \in \mathcal{F}$ and all $\pi \in \mathcal{G}$, $\pi H \pi^{-1} \in \mathcal{F}$.

Let $\mathcal{G}$ and a normal filter $\mathcal{F}$ on $\mathcal{G}$ be fixed. We say that a name $x \in \mathcal{M}^B$ is *symmetric* if $\mathrm{sym}(x) \in \mathcal{F}$. The class HS of all *hereditarily symmetric* names is defined by recursion:

1. $\emptyset \in \mathrm{HS}$.
2. If $\mathrm{dom}(x) \subset \mathrm{HS}$ and x is symmetric, then $x \in \mathrm{HS}$.

It is clear that HS contains all the canonical names of elements of $\mathcal{M}$.

Now let G be an $\mathcal{M}$-generic ultrafilter on B, and let i_G be the interpretation function of $\mathcal{M}^B$ by G. Then we define

$$\mathcal{N} = \{i_G(x) : x \in \mathrm{HS}\}.$$

It is easy to see that $\mathcal{M} \subset \mathcal{N} \subset \mathcal{M}[G]$. The proof of the following fundamental theorem can be found in Jech [Jec73].

Theorem 31. *The class $\mathcal{N}$, constructed as above, is a model of ZF.* □

We call $\mathcal{N}$ a *symmetric extension* of $\mathcal{M}$, or a *symmetric model.*

The Basic Cohen Model

We construct the model of ZF known as the Basic Cohen Model (BCM) or Cohen-Halpern-Lévy Model, as a symmetric model.

Let $\mathcal{M}$ be a model of ZFC, and consider the following forcing notion:

$$\mathbb{P} = \{p : p \text{ is a finite function, } \mathrm{dom}(p) \subset \omega \times \omega, \text{ and } \mathrm{range}(p) \subset \{0,1\}\},$$

with the partial order given by reverse inclusion. Let B be the complete Boolean algebra $\mathrm{RO}(\mathbb{P})$ of regular open sets in $\mathbb{P}$ with the partial order topology.

Let G be a generic ultrafilter on B. For each $n \in \omega$, let

$$x_n = \{m \in \omega : \text{there exists } p \in G \text{ such that } p(n,m) = 1\}$$

and let $A = \{x_n : n \in \omega\}$. The sets $x_n, n \in \omega$ have canonical names $\dot{x}_n$ defined such that for all $m \in \omega$, $\dot{x}_n(\check{m}) = \sup_B\{p \in \mathbb{P} : p(n,m) = 1\}$.

Now, let π be any permutation of ω. We can extend π to an order preserving bijection from $\langle \mathbb{P}, \leq \rangle$ onto itself as follows:

$$\mathrm{dom}(\pi p) = \{\langle \pi n, m\rangle : \langle n, m\rangle \in \mathrm{dom}(p)\}$$

and

$$(\pi p)(\pi n, m) = p(n,m).$$

In turn, this π induces an automorphism of B by

$$\pi u = \sup\{\pi p : p \leq u\}.$$

Take $\mathcal{G}$ to be the group of all automorphisms of B which are induced by permutations of ω as described above, and take $\mathcal{F}$ to be the filter of subgroups of $\mathcal{G}$ generated by the set $\{\text{fix}(e) : e \text{ is a finite subset of } \omega\}$, where

$$\text{fix}(e) = \{\pi \in \mathcal{G} : \pi n = n \text{ for all } n \in e\}.$$

It is easy to check that $\mathcal{F}$ is a normal filter on $\mathcal{G}$. Let $\mathcal{N}$ be the symmetric extension of $\mathcal{M}$ obtained using $\mathcal{G}$ and $\mathcal{F}$. This is the model we call BCM.

The following lemma is a well known result:

Lemma 32. *The set $A \in \mathcal{N}$ is Dedekind-finite.* □

Some Properties of BCM

The model BCM has been well studied, and there are many consequences of AC that have been proved to hold in it. One of them (that can actually be derived from the Boolean Prime Ideal Theorem, which holds in BCM), is the following:

Theorem 33. *The Linear Order Principle (that states that every set can be linearly ordered) holds in BCM.* □

We will sketch here part of the development of the proof of the previous theorem (the full proof can be found in Jech [Jec73]), since we will use this machinery to prove our main result about BCM.

We say that $e \subset \omega$ is a support for a name $\dot{x}$ if $\text{fix}(e) \subset \text{sym}(\dot{x})$. It is not too difficult to check that the intersection of two supports for a name is also a support for that name; therefore, for each name $\dot{x}$ there exists a least support $s(\dot{x})$. Notice that if $\pi \restriction s(\dot{x}) = \rho \restriction s(\dot{x})$, then $\pi(\dot{x}) = \rho(\dot{x})$, since $\rho^{-1}\pi \in \text{fix}(s(\dot{x}))$.

An *assignment* t is a one-to-one function whose domain is a finite subset of ω and whose values are elements of A. Let t be an assignment,

$$t(n_1) = x_{i_1}, \ldots, t(n_k) = x_{i_k},$$

and let $\dot{x} \in \mathrm{HS}$ be such that $\{n_1, \ldots, n_k\} \supset s(\dot{x})$. Then, if π is any permutation such that

$$\pi(n_1) = i_1, \ldots, \pi(n_k) = i_k,$$

then $\pi(\dot{x})$ is unique regardless of the behavior of π outside of $\{n_1, \ldots, n_k\}$ (see the last comment in the previous paragraph). Define then

$$F(t, \dot{x}) = i_G(\pi\dot{x}).$$

The following lemma is proved in Jech [Jec73]:

Lemma 34. *The function F is in $\mathcal{N}$.* □

Notice that whenever t is an assignment and $\dot{x}$ is a name such that $|t| \geq |s(\dot{x})|$, then there is $\dot{y} \in \mathrm{HS}$ such that $i_G(\dot{x}) = F(t, \dot{y})$; actually, this name $\dot{y}$ is just $\pi\dot{x}$ for some $\pi \in \mathcal{G}$.

Lemma 35. *If t is an assignment and $z \in x = F(t, \dot{y})$, then there exist an assignment $t' \supset t$ and a name $\dot{z} \in \mathrm{dom}(\dot{y})$ such that $z = F(t', \dot{z})$.*

Proof. Let π be a permutation of ω such that

$$t(n_1) = x_{\pi(n_1)}, \ldots, t(n_k) = x_{\pi(n_k)}.$$

Since $x = i_G(\pi\dot{y})$, there exists $\dot{z}' \in \mathrm{dom}(\pi\dot{y})$ such that $z = i_G(\dot{z}')$. Taking t' so that it contains t, agrees with π, and its domain contains a support of $\dot{z} = \pi^{-1}\dot{z}'$, we have that $F(t', \dot{z}) = z$. □

Now we can prove the main property of BCM for us in this section:

Theorem 36. *C^{-}(VII) holds in BCM.*

Proof. Let X be a non-well-orderable set, and let $\dot{X}$ be a name for X. We want to show that there is an infinite subset of X with a choice function.

For each $x \in X$, there is some $\dot{y} \in \mathrm{dom}(\dot{X})$ and an assignment t such that $x = F(t, \dot{y})$. If for all names $\dot{y} \in \mathrm{dom}(\dot{X})$ the set $\{x \in X : \text{there exists } t \text{ such that } x = F(t, \dot{y})\}$ is finite, then we can use the fact that BCM satisfies the Linear Ordering Principle, and the fact that $\mathrm{dom}(\dot{X})$ is well orderable, since it is in the ground model, to find a well order for X.

So, there exists at least one name $\dot{y}_0$ such that

$$\{x \in X : \text{there exists } t \text{ such that } x = F(t, \dot{y}_0)\}$$

is infinite. Using Lemma 35, choose $\dot{z} \in \mathrm{dom}(\dot{y}_0)$, and assignments t_0, t_1 of minimum size such that

1. $\mathrm{dom}(t_0) = s(\dot{y}_0)$,
2. $F(t_0, \dot{y}_0) \in X$,
3. $t_0 \subset t_1$, and
4. $F(t_1, \dot{z}) \in F(t_0, \dot{y}_0)$.

Consider the set T of assignments given by

$$T = \{t : \mathrm{dom}(t) = \mathrm{dom}(t_1), t \supset t_1 \setminus t_0, F(t, \dot{y}_0) \in X\}.$$

Then we claim that $g = \{\langle F(t, \dot{y}_0), F(t, \dot{z})\rangle : t \in T\}$ is a choice function with infinite domain contained in X.

Indeed, g is a function, since if $F(t, \dot{y}_0) = F(t', \dot{y}_0)$, that is,

$$i_G(\pi \dot{y}_0) = i_G(\pi' \dot{y}_0),$$

where π agrees with t and π' agrees with t', then the names $\pi \dot{y}_0$ and $\pi' \dot{y}_0$ have the same minimum support. Consequently, $t \restriction \mathrm{dom}(t_0) = t' \restriction \mathrm{dom}(t_0)$, and that implies that $t \restriction \mathrm{dom}(t_1) = t' \restriction \mathrm{dom}(t_1)$. Therefore, $F(t, \dot{z}) = F(t', \dot{z})$.

Also, $\mathrm{dom}(g)$ is infinite, by our choice of $\dot{y}_0$. Finally, for all adequate t,

$$F(t, \dot{z}) = i_G(\pi\dot{z}) \in i_G(\pi\dot{y}_0) = F(t, \dot{y}_0).$$

$\square$

In Lemma 32 we saw that there exists an infinite IV-finite set in BCM. Therefore, by Theorem 15, we conclude that $C(\mathrm{IV})$ is false in BCM. This proves the following result:

Theorem 37. *In ZF it cannot be proved that $C^-(VII)$ implies $C(IV)$.* $\square$

Another conclusion is obtained using the following well known result:

Lemma 38. *$E(I, III)$ holds in BCM.* $\square$

Since $E(\mathrm{I}, \mathrm{III})$ implies $C(\mathrm{III})$, we get:

Corollary 39. *In ZF it cannot be proved that $C(III)$ implies $C(IV)$.* $\square$

1.6 The Embedding Theorem

In this section we use Jech and Sochor's embedding theorem in an unorthodox way to obtain several results.

Given a set X, we define $\mathcal{P}^\alpha(X)$ for every ordinal α by recursion:

1. $\mathcal{P}^0(X) = X$,
2. $\mathcal{P}^{\nu+1}(X) = \mathcal{P}(\mathcal{P}^\nu(X))$, and
3. $\mathcal{P}^\lambda(X) = \bigcup_{\nu<\lambda} \mathcal{P}^\nu(X)$.

Now we state the embedding theorem:

Theorem 40 (Jech-Sochor). *Let $\mathcal{U}$ be a model of ZFA + AC, let A be the set of atoms of $\mathcal{U}$, let $\mathcal{M}$ be the kernel (the class of pure sets) of $\mathcal{U}$, and let α be an ordinal in $\mathcal{U}$. Then, for every permutation model $\mathcal{V} \subset \mathcal{U}$ (a model of ZFA), there exists a symmetric extension $\mathcal{N} \supset \mathcal{M}$ (a model of ZF) and a set $\widetilde{A} \in \mathcal{N}$ such that*

$$(\mathcal{P}^\alpha(A))^\mathcal{V} \text{ is } \in\text{-isomorphic to } (\mathcal{P}^\alpha(\widetilde{A}))^\mathcal{N}.$$

For a full proof, refer to Jech [Jec73]. Here we will describe the construction given there, and prove a property that is key for our purposes.

The Construction

The idea behind the construction is the following: using forcing, we will add to the kernel $\mathcal{M}$ (which is a model of ZFC) one generic set $\tilde{a}$ of subsets of an ordinal κ for each $a \in A$; the set $\widetilde{A} = \{\tilde{a} : a \in A\}$ plays the role of the set of atoms.

The reason why we use sets of sets of ordinals instead of simply using sets of ordinals, is that sets of ordinals have non-trivial relations between them; for example, the set of subsets of an ordinal is linearly ordered. For the same reason, there should be no choice function for the set $\widetilde{A}$, otherwise the structure from the sets of ordinals would be lifted up to the elements of $\widetilde{A}$. Also, we must make sure that the objects that we use to represent the atoms will not appear again in $(\mathcal{P}^\alpha(\widetilde{A}))^\mathcal{N}$; for that reason we take κ large enough so that the sets $\tilde{a}$ do not appear in the first α levels.

Let α be an ordinal in $\mathcal{U}$, let $\mathcal{G}, \mathcal{F} \in \mathcal{U}$ be a group of permutations of the set A of all atoms of $\mathcal{U}$ and a normal filter on $\mathcal{G}$, respectively. Let $\mathcal{V}$ be the permutation model determined by $\mathcal{G}, \mathcal{F}$, and let $\mathcal{M}$ be the kernel of $\mathcal{U}$. We will first construct a generic extension $\mathcal{M}[G]$.

Let κ be a regular cardinal such that $\kappa > |\mathcal{P}^\alpha(A)|$ (in $\mathcal{U}$). The set $\mathbb{P}$ of forcing conditions consists of functions p with values in $\{0,1\}$ such that $|\operatorname{dom}(p)| < \kappa$ and $\operatorname{dom}(p) \subset (A \times \kappa) \times \kappa$, ordered by reverse inclusion.

Let G be an $\mathcal{M}$-generic ultrafilter on $B = RO(\mathbb{P})$. We define the following elements of $\mathcal{M}[G]$, together with their canonical names:

1. $x_{a\xi} \subset \kappa$, $\eta \in x_{a\xi}$ iff there exists $p \in G$ such that $p(a,\xi,\eta) = 1$, for $a \in A, \xi < \kappa$.
2. $\operatorname{dom}(\dot{x}_{a\xi}) = \{\check{\eta} : \eta < \kappa\}$, $\dot{x}_{a\xi}(\check{\eta}) = \sup\{p \in \mathbb{P} : p(a,\xi,\eta) = 1\}$.
3. $\tilde{a} = \{x_{a\xi} : \xi < \kappa\}$, for each $a \in A$.
4. $\dot{\tilde{a}}(\dot{x}_{a\xi}) = 1$, for all $\xi < \kappa$.
5. $\widetilde{A} = \{\tilde{a} : a \in A\}$.

6. $\dot{\tilde{A}}(\dot{a}) = 1$ for all $a \in A$.

For every $x \in \mathcal{U}$ we define $\tilde{x} \in \mathcal{M}[G]$ and its canonical name $\dot{\tilde{x}}$ by $\in$-recursion. If x is a set in $\mathcal{U}$ and $\tilde{y}, \dot{\tilde{y}}$ have been defined for every $y \in x$, we define

$$\tilde{x} = \{\tilde{y} : y \in x\}$$

and

$$\mathrm{dom}(\dot{\tilde{x}}) = \{\dot{\tilde{y}} : y \in x\}, \quad \dot{\tilde{x}}(\dot{\tilde{y}}) = 1, \text{ for all } y \in x$$

We construct now a symmetric submodel $\mathcal{N}$ of $\mathcal{M}[G]$. For that we need to define a group $\mathcal{G}'$ of automorphisms of B as well as a normal filter $\mathcal{F}'$ on $\mathcal{G}'$.

For every permutation ρ of A, let $[\rho]$ be the set of all permutations π of $A \times \kappa$ such that for all a, ξ,

$$\pi(a, \xi) = (\rho(a), \xi'), \quad \text{for some } \xi'.$$

We define then

$$\mathcal{G}' = \bigcup\{[\rho] : \rho \in \mathcal{G}'\}.$$

Similarly, we define $H' = \bigcup\{[\rho] : \rho \in H\}$ for every subgroup H of $\mathcal{G}$.

We extend the permutations π of $A \times \kappa$ to automorphisms of $\mathbb{P}$ by defining, for each $p \in \mathbb{P}$, $(\pi p)(\pi(a, \xi), \eta) = p(a, \xi, \eta)$ for all $\langle a, \xi, \eta\rangle \in \mathrm{dom}(p)$. Then π can be easily extended to the Boolean algebra B. This way we can consider $\mathcal{G}'$ as a group of automorphisms of B.

The normal filter $\mathcal{F}'$ is obtained as the filter generated by the set of subgroups

$$\{H' : H \in \mathcal{F}\} \cup \{\mathrm{fix}(e) : e \subset A \times \kappa,\ e \text{ finite}\}.$$

Let $\mathcal{N}$ be the symmetric model determined by $\mathcal{G}'$ and $\mathcal{F}'$. This is the model that satisfies the conditions in the statement of Theorem 40.

A Key Lemma and Some Results

The following lemma establishes that an expected property of the set $\widetilde{A}$ does indeed hold.

Lemma 41. *Let $\mathcal{N}$ be the Jech-Sochor transfer of a permutation model $\mathcal{V}$, as defined in the previous section, with $\widetilde{A}$ being the set in correspondence with the set A of atoms in $\mathcal{V}$. Then there is no infinite subset of $\widetilde{A}$ with a choice function.*

Proof. Suppose that f is a choice function for some infinite subset of $\widetilde{A}$. Let $\dot{f}$ be a name for f and let p be a condition that forces:

1. $\dot{f}$ is a function,
2. $\operatorname{dom}(\dot{f}) \subset \dot{\widetilde{A}}$ is infinite,
3. $\dot{f}(x) \in x$ for all $x \in \operatorname{dom}(\dot{f})$.

Let also $e \subset A \times \kappa$ be a finite set such that $\operatorname{fix}(e) \subset \operatorname{sym}(\dot{f})$.

Since p forces that $\operatorname{dom}(\dot{f})$ is infinite, then p forces that there exists $\tilde{a} \in \widetilde{A}$ such that $a \notin e$. Take $q \leq p$, $a \in A$, and $x_{a\xi} \in \tilde{a}$ such that

$$q \Vdash \check{a} \times \check{\kappa} \cap \check{e} = \check{\emptyset} \text{ and } \dot{f}(\dot{\tilde{a}}) = \dot{x}_{a\xi},$$

where $\check{a}$ and *checke* are canonical names for a and e, respectively, or for some elements in the kernel that represent them. Take η such that $\langle a, \eta, \gamma \rangle \notin \operatorname{dom}(q)$ for all $\gamma < \kappa$. Then the permutation π that sends

$$\pi : \langle a, \eta \rangle \mapsto \langle a, \xi \rangle \mapsto \langle a, \eta \rangle$$

and fixes everything else is an element of $\mathcal{G}'$. Also,

$$\operatorname{dom}(\pi q) = (\operatorname{dom}(q) \smallsetminus \{\langle a, \xi, \gamma \rangle : \gamma < \kappa\}) \cup \{\langle a, \eta, \gamma \rangle : \langle a, \xi, \gamma \rangle \in \operatorname{dom}(q)\},$$

so q and πq are compatible. Let r be a common extension. Then r forces (1), (2), and (3), it forces $\dot{f}(\dot{\tilde{a}}) = \dot{x}_{a\xi}$, but also forces $(\pi\dot{f})(\pi\dot{\tilde{a}}) = \pi\dot{x}_{a\xi}$, that is, $\dot{f}(\dot{\tilde{a}}) = \dot{x}_{a\eta}$. Since we have $\Vdash \dot{x}_{a\eta} \neq \dot{x}_{a\xi}$, we get a contradiction with $r \Vdash (1)$. $\square$

Theorem 40 is commonly used through a lemma like the following:

Lemma 42. *Let ϕ be a formula of the form $\exists\nu\psi(X,\nu)$, where the only quantifiers allowed in ψ are of the form $\exists u \in \mathcal{P}^{\nu}(X)$ and $\forall u \in \mathcal{P}^{\nu}(X)$. If $\mathcal{V}$ is a permutation model such that $\mathcal{V} \models \exists X\phi(X)$, then there exists a symmetric model $\mathcal{N}$ of ZF such that $\mathcal{N} \models \exists X\phi(X)$.*

A sentence like $\exists X\phi(X)$ is called *boundable.* Lemma 42 establishes that every boundable sentence is *transferable*, that is, if it is satisfied in a permutation model, there exists a Jech-Sochor symmetric model that also satisfies it. A few well known boundable sentences are the following:

1. $\exists X(X$ is amorphous).
2. $\exists X(X$ is an infinite II-finite set).
3. $\exists X(X$ is an infinite III-finite set).
4. $\exists X(X$ is an infinite IV-finite set).
5. $\exists X(X$ is an infinite Δ_5-finite set).

The examples above, from (1) to (4), appear in Jech [Jec73]. For (5), notice that if there is a surjection f from a set X onto $X \cup \{X\}$, and the rank of X is γ, then the rank of f is at most $\gamma+3$, and $f \in \mathcal{P}^3(X)$. Then the statement "X is Δ_5-finite" can be written as

$$\begin{aligned}\forall f \in & \mathcal{P}^3(X)\,(\exists y \in X \cup \{X\}, \forall x \in X(\langle x,y\rangle \notin f) \vee \\ & (\exists y,y' \in X \cup \{X\}, \exists x \in X(\langle x,y\rangle \in f \wedge \langle x,y'\rangle \in f)))\,.\end{aligned}$$

More general classes of formulas are transferable. In particular, so-called *injectively* and *surjectively boundable* statements are transferable; see Pincus [Pin72] for the definitions and a thorough discussion of the subject. We will not state those transfer theorems here for lack of space. However we will mention what results are being used, and the relevant references.

As a consequence of the previous lemmas, we obtain the following:

Theorem 43. *None of the following statements is a theorem in ZF:*

1. *$C(Ia)$ implies $C^-(II)$,*
2. *$C(II)$ implies $C^-(III)$,*
3. *$C(III)$ implies $C^-(\Delta_5)$,*
4. *$C(IV)$ implies $C^-(V)$.*

Proof. All the four parts will be proved in a similar way. In order to obtain a model of ZF where $C(\mathcal{Q})$ holds but $C^-(\mathcal{Q}')$ fails, we start with a permutation model $\mathcal{V}$ where $E(I, \mathcal{Q})$ holds (and in consequence $C(\mathcal{Q})$ holds too), while A is an example of an infinite $\mathcal{Q}'$-finite set. Then we take the Jech-Sochor transfer model $\mathcal{N}$; in this case the set $\widetilde{A}$ is an example of an infinite $\mathcal{Q}'$-finite set without a partial choice function (by Lemma 41). The only detail is to show that $E(I, \mathcal{Q})$ still holds in $\mathcal{N}$.

For (1), consider the permutation model MPM-I, Mathias-Pincus Model I. In the Jech-Sochor transfer of MPM-I there are no amorphous sets (for the transferability of this statement, see Pincus [Pin72], 2B3), so $C(\mathrm{Ia})$ holds, but $\widetilde{A}$ is a II-finite family without partial choice functions.

For (2), consider MOM, Mostowski's ordered model. In the Jech-Sochor transfer of MOM we have $E(\mathrm{I}, \mathrm{II})$ (for the transferability, see Pincus [Pin72], 2B4), so $C(\mathrm{II})$ holds. However, $\widetilde{A}$ is a III-finite family without a partial choice function.

For (3), take SFM, Fraenkel's second model. $E(\mathrm{I}, \mathrm{III})$ transfers because it is surjectively boundable, so it holds in the Jech-Sochor transfer $\mathcal{N}$ of SFM. Therefore $C(\mathrm{III})$ holds in $\mathcal{N}$. Also, $\widetilde{A}$ is a Δ_5-finite family without a partial choice function.

Finally, for (4), consider the modified Fraenkel model $\mathrm{BFM}(\aleph_1)$. The statement $E(\mathrm{I}, \mathrm{IV})$ transfers since it is an injectively bounded sentence. Therefore, $C(\mathrm{IV})$ holds in the Jech-Sochor transfer of $\mathrm{BFM}(\aleph_1)$. Of course, $\widetilde{A}$ is a V-finite family without a partial choice function. □

CHAPTER 2
COUNTABLE COMPACT SPACES
IN THE ABSENCE OF AC

2.1 Introduction

It is a well known result that a compact Hausdorff space with no isolated points has cardinality greater than or equal to the continuum (see, for example, Sierpiński [Sie52], Theorem 60). The known proofs use the axiom of choice (AC), or at least make extra assumptions about the space that imply the existence of well ordered local bases. Morillon posed the question of whether this result can be proved without any use of AC (see Miller [Mil93]). Here we answer that question negatively, by constructing a model of ZF where there exists a compact Hausdorff topology on ω with no isolated points.

The topology above turns out to be extremally disconnected, and this leads into another question posed by Morillon (personal communication, 1998): Can it be proved from ZF alone that every connected compact Hausdorff space with at least two elements has cardinality at least $2^{\aleph_0}$? The answer, which constitutes the main result of this chapter, is also negative, and, of course, it subsumes the answer to the first question. However, we will include the construction of the first model because it is much simpler and easier to deal with, and because it offers a preview of some techniques needed for the main result. The more elaborate construction of the second model, in which there is a connected compact Hausdorff topology on ω, will be done in Section 2.4.

The next section contains the original result, and a few modifications that can be proved in ZF. These modified versions depend on the use of a stronger (namely, *effective*) version of the notion of compactness.

2.2 The Original Result and Some Variations

The original result is the following:

Theorem 1 (AC). *If X is a compact Hausdorff space with no isolated points, then* $|X| \geq 2^{\aleph_0}$.

We will not provide the (easy and well known) proof here. Instead, we will obtain the result as an immediate consequence of Corollary 7.

Definition 2. Let $\langle X, \tau \rangle$ be a topological space. We say that X is *effectively compact* if there exists a function Σ that assigns to each open cover C of X a finite subcover $\Sigma(C)$.

Clearly, every effectively compact space is compact. Of course, if AC holds, then every space is effectively compact if and only if it is compact.

Our next lemma establishes that every effectively compact, Hausdorff space is "effectively regular."

Lemma 3. *Let $\langle X, \tau \rangle$ be an effectively compact, Hausdorff space. Then there is a function Γ that maps every pair $\langle x, S \rangle$, where S is a closed non-empty set and $x \in X \setminus S$, to a pair $\langle A, B \rangle$ of disjoint open sets such that $x \in A$ and $S \subset B$.*

Proof. Let $x \in X$ and let S be a closed subset of X such that $x \notin S$. Also, let Σ be the function that witnesses the fact that X is effectively compact; we will use Σ to find a uniquely determined pair of separating neighborhoods for x, S.

Consider the collection

$$C = \{W \in \tau : \text{there exists } W' \in \tau \text{ s.t. } W \cap W' = \emptyset \text{ and } x \in W'\} \cup \{S^c\},$$

where $S^c = X \setminus S$. Clearly, C is an open cover for X, since X is Hausdorff and therefore every element of S can be separated from x by open neighborhoods. Let C' be the finite subcover of C determined by Σ, and let $B = \bigcup(C' \setminus \{S^c\})$; it is easy to see that $S \subset B$ and that $x \notin B$.

We claim that x is not in the closure of B. Indeed, if for each $W \in C' \setminus \{S^c\}$ we choose a neighborhood W' of x disjoint from W, we have that the intersection of all these neighborhoods W' is an open neighborhood of x which is disjoint from B (notice that we performed only finitely many arbitrary choices to obtain the neighborhoods W', and that these choices do not affect the final definition of A). We can now define A as the complement of the closure of B. □

This lemma implies, in particular, that every effectively compact, Hausdorff space is also "effectively Hausdorff." It can also be proved, by modifying the proof above slightly, that every effectively compact, Hausdorff space is also "effectively normal;" however, that result will not be needed here. Instead, we will prove the lemma below. Given a set A, $\bar{A}$ stands for the topological closure of A.

Lemma 4. *Let $\langle X, \tau \rangle$ be an effectively compact, Hausdorff space. Then there is a function Δ that maps every pair $\langle E, F \rangle$ of non-empty finite subsets of X to a pair $\langle A, B \rangle$ of open sets such that $E \subset A$, $F \subset B$, and $\bar{A} \cap \bar{B} = \emptyset$.*

Proof. Let $\langle E, F \rangle$ be a pair of finite subsets of X, and let Γ be as in Lemma 3; we will use Γ to find a uniquely determined pair of separating neighborhoods for E, F whose closures are disjoint, and define that pair as the value of $\Delta(E, F)$

First we will prove that for every pair $\langle x, y \rangle$ of elements from X there exist uniquely determined neighborhoods $A_{x,y}$ of x and $B_{x,y}$ of y which have disjoint closures. Let $\langle A_1, B_1 \rangle = \Delta(x, \{y\})$. Clearly $x \notin \bar{B}_1$; let $\langle A_2, B_2 \rangle = \Delta(x, \bar{B}_1)$. Then $\bar{A}_2 \subset X \setminus B_2$, and consequently $\bar{A}_2 \cap \bar{B}_1 = \emptyset$. Therefore we can take $A_{x,y} = A_2$ and $B_{x,y} = B_1$.

Now, for each element $x \in E$, define $A_x = \bigcap_{y \in F} A_{x,y}$, and for each element $y \in F$ define $B_y = \bigcap_{x \in E} B_{x,y}$. Defining $A = \bigcup_{x \in E} A_x$ and $B = \bigcup_{y \in F} B_y$ we can easily check that A and B satisfy the required properties. □

The following lemma puts some restrictions on the kinds of sets that can be furnished with an effectively compact, Hausdorff topology.

Lemma 5. *Let $\langle X, \tau \rangle$ be an effectively compact, Hausdorff space. Then there is a multiple choice function Λ for $\mathcal{P}(X) \smallsetminus \{\emptyset\}$, that is, a function that chooses a finite non-empty subset out of every non-empty subset of X. Also, if X has a well orderable dense subset, then there is a choice function for $\mathcal{P}(X) \smallsetminus \{\emptyset\}$, and therefore X is well orderable.*

Proof. Given a non-empty subset $Y \subset X$, we use the function Σ to define a unique finite subset $E \subset Y$.

If Y is a singleton, we can choose $E = Y$. So, assume that Y has at least two elements. In this case the collection $C = \{X \smallsetminus \{y\} : y \in Y\}$ is an open cover for X. Then we can define

$$E = \{y \in Y : X \smallsetminus \{y\} \in \Sigma(C)\}.$$

For the second part of the lemma, assume that X has a well ordered dense subset $Z = \{z_\alpha : \alpha < \lambda\}$. We will find a uniquely defined element $y_0 \in Y$.

If E, found as above, is a singleton, we can take y_0 to be the only element of E. So, assume that E has at least two elements. For each $y \in E$, define $A_y = \bigcap_{y' \in E \smallsetminus \{y\}} \Delta(\{y\}, \{y'\})_1$, where $\Delta(F, F')_1$ stands for the neighborhood of containing F. This way, A_y is a neighborhood of y, for each $y \in E$, and if y, y' are different, then $A_y \cap A_{y'} = \emptyset$. This way, to each $y \in E$ we can assign a finite set of ordinals $D_y = \{\alpha : z_\alpha \in A_y\}$; since the sets D_y are pairwise disjoint, we can canonically choose y_0 to be such that $\min(D_{y_0})$ is minimum. $\square$

A consequence of the previous lemma is that if the space $[0, 1]$ is effectively compact, then $\mathbb{R}$ is well orderable.

Now we can prove the main result in this section.

Theorem 6. *Let X be an effectively compact, Hausdorff space, and assume that X has no isolated points. Then $|X| \geq^* 2^{\aleph_0}$ (see Appendix A for the definition of $\geq^*$).*

Proof. Let X be an effectively compact, Hausdorff space with no isolated points. Also, let Σ be a function that chooses finite subcovers from each open cover, Δ a function that chooses separating neighborhoods with disjoint closures for every pair $\langle E, F\rangle$ of non-empty finite subsets of X, and Λ a multiple choice function for $\mathcal{P}(X) \setminus \{\emptyset\}$ (as given by the definition of effective compactness, Lemma 4, and Lemma 5, respectively).

To each element s of $2^{<\omega}$ we assign a finite set $E_s \subset X$ and a neighborhood U_s containing E_s by induction on the length of s. First, define $E_\emptyset = \Lambda(X)$ and $U_\emptyset = X$. If E_s and U_s are defined, we define:

1. $E_{s^\frown 0} = E_s$.
2. $E_{s^\frown 1} = \Lambda(U_s \setminus \{E_s\})$ (since there are no isolated points, every open set is infinite, so $U_s \setminus \{E_s\}$ is not empty).
3. If $\Delta(E_{s^\frown 0}, E_{s^\frown 1}) = \langle U, U'\rangle$, then define $U_{s^\frown 0} = U \cap U_s$ and $U_{s^\frown 1} = U' \cap U_s$

It is clear that if s is an initial segment of s', then $E_{s'} \subset U_s$, and that if neither s nor s' is an initial segment of the other, then $\bar{U}_s \cap \bar{U}_{s'} = \emptyset$.

For every $\sigma \in {}^\omega 2$ we define the set

$$A_\sigma = \bigcap_{m\in\omega} \bar{U}_{\sigma\restriction m}.$$

The following things can be proved about the sets $A_\sigma, \sigma \in {}^\omega 2$:

1. Each A_σ is not empty: Since each set $\bar{U}_{\sigma\restriction m}$ is non-empty, the intersection is non-empty (otherwise the set of complements would be an open cover without a finite subcover).
2. If $\sigma \neq \rho$, then $A_\sigma \cap A_\rho = \emptyset$: Take m large enough such that $\sigma \restriction n \neq \rho \restriction n$. Then $\bar{U}_{\sigma\restriction n} \cap \bar{U}_{\rho\restriction n} = \emptyset$, and we have that $A_\sigma \subset \bar{U}_{\sigma\restriction n}$ and $A_\rho \subset \bar{U}_{\rho\restriction n}$

As a consequence of these two facts we can conclude that $|X| \geq^* 2^{\aleph_0}$: we can define a surjection $f\colon X \to 2^\omega$ by assigning $f(x) = \sigma$ for all $x \in A_\sigma$, for each $\sigma \in 2^\omega$, and assigning all other elements of X to a fixed element of 2^ω. $\square$

We could prove the stronger inequality $|X| \geq 2^{\aleph_0}$, if we were able to choose one element from each set A_σ and define this way a one-to-one function from 2^ω into X. This can be done if, for example, if X is well orderable or, equivalently, if X has a well orderable dense subset (see Lemma 5). In that case we obtain the following corollary:

Corollary 7. *If X is an effectively compact, Hausdorff space with a well orderable dense subset and no isolated points, then $|X| \geq 2^{\aleph_0}$. If there is such a space X, then it is well orderable, and consequently $2^{\aleph_0}$ is a well orderable cardinal.*

From this result we can easily obtain the original result, Theorem 1, by noticing that, under AC, compactness is equivalent to effective compactness and every space is well orderable. As a final consequence, we obtain another corollary:

Corollary 8. *If X is a countable, effectively compact, Hausdorff space, then X has an isolated point.*

2.3 A Simple Model

In this section we describe a model of ZF where there exists a compact Hausdorff topology on ω with no isolated points. We will skip some of the proofs, since the same techniques will be applied in Section 2.4.

We begin with a standard model $\mathcal{M}$ of ZFC. Using Cohen forcing, we add a countable set $A = \{x_n : n \in \omega\}$ of mutually generic subsets of ω, and define B as the set of finite Boolean combinations of elements of A (using unions and complements with respect to ω). Since the generic extension $\mathcal{M}[G]$ satisfies ZFC, B is isomorphic (in $\mathcal{M}[G]$) to the countable free Boolean algebra.

We define our model as the class

$$\mathcal{N} = \mathrm{HOD}^{\mathcal{M}[G]}(B \cup \{B\})$$

of sets that are hereditarily ordinal definable from $B \cup \{B\}$ in $\mathcal{M}[G]$; that is, $\mathcal{N}$ is formed by all sets $x \in \mathcal{M}[G]$ such that x and all the elements of the transitive closure of x are definable by formulas in the language of set theory (relativized to $\mathcal{M}[G]$), using as parameters only ordinals, elements of B, or B itself. It is a well known result that such a class is a model of ZF.

Clearly, $B \in \mathcal{N}$, and it is still an atomless Boolean algebra of subsets of ω in $\mathcal{N}$. However, the isomorphism between B and the countable free Boolean algebra is *not* in $\mathcal{N}$; in fact, B is not even well orderable in $\mathcal{N}$. Even more, the generating set A is not in $\mathcal{N}$ either, nor is its enumeration by ω. We define the topology τ on ω as the topology generated by B as a base.

It is easy to check that $\langle \omega, \tau \rangle$ is Hausdorff: given $m, m' \in \omega$, with $m \neq m'$, standard forcing arguments can be used to show that there exists $x_n \in A$ such that $m \in x_n$ and $m' \notin x_n$. Therefore, x_n and its complement, which are elements of B, are separating neighborhoods for m and m'. It is even easier to see that $\langle \omega, \tau \rangle$ has no isolated points, since B is atomless and therefore every non-empty basic neighborhood is infinite.

In order to prove that $\langle \omega, \tau \rangle$ is compact, it is enough to check that every cover C of ω by elements of B has a finite subcover. We will assume first that C is definable by a formula that uses ordinals and B as parameters, but not members of B; in this case we use the following lemma, whose proof we will skip:

Lemma 9. *Let $C \subset B$ be definable by a formula that uses ordinals and B as parameters, but not members of B. Then there exists a set $D \subset 2^{<\omega}$ such that for all $x \in B$,*

$x \in C$ iff there exists $d \in D$ such that $\chi_x \supset d$,

where χ_x is the characteristic function of x. $\square$

Assume now that $C \subset B$ is a cover for ω definable from ordinals and B, and let $D \subset 2^{<\omega}$ be the set corresponding to C obtained from Lemma 9. Pick $x \in C$, and $d \in D$ such that $d \subset \chi_x$; we can assume without loss of generality that $\mathrm{dom}(d) = \{0, \ldots, u-1\}$. Since C is a cover for ω, we can choose $y_0, \ldots, y_{v-1} \in C$ such that $\mathrm{dom}(d) \subset \bigcup\{y_i : i = 0, \ldots, v-1\}$.

We would like the complement $\bar{x}$ of x to be in C, but that might fail. So, we modify $\bar{x}$ into a set $w \in B$ in such a way to guarantee that the modified version belongs in C. However, to compensate for those modifications, we will have to perform the opposite modifications to x, in such a way that the result w' contains the complement of w but is still in C.

Let $d_0, \ldots, d_{u-1}$ be finite functions into $\{0,1\}$ with domain $\{0, \ldots, u-1\}$, defined by $d_i(j) = 1$ iff $i = j$, for $i, j \in \{0, \ldots, u-1\}$. Also, choose sets $z_0, \ldots, z_{u-1}$ such that $d_i \subset \chi_{z_i}$, for $i = 0, \ldots, u-1$. Define

$$w = \left(\bar{x} \setminus \bigcup\{z_i : d(i) = 0\}\right) \cup \bigcup\{z_i : d(i) = 1\}$$

and

$$w' = x \cup \bigcup\{z_i : d(i) = 0\}.$$

It is clear that both w and w' are in C, since both extend d. Also, it is easy to see that $w \cup w'$ covers all of ω except for those numbers $i \in \omega$ for which $d(i) = 0$. Therefore, the finite collection $C' = \{w, w', y_0, \ldots, y_{v-1}\}$ covers all of ω.

For the case in which the definition of C depends on parameters $w_0, \ldots, w_{k-1}$, we consider the Boolean subalgebra B' of B generated by $\{w_0, \ldots, w_{k-1}\}$. Since B' is finite, it is atomic; let $a_0, \ldots, a_{l-1}$ be the atoms of B'. Then C induces covers $C_i = \{x \cap a_i : x \in C\}$ for each a_i, $i = 0, \ldots, l-1$. Both Lemma 9 and the argument above can be relativized to each set a_i, in order to obtain a finite subcover C_i' of C_i,

for $i = 0, \ldots, l-1$. These finitely many finite covers can be used to obtain a finite subset C' of C which covers ω, since $\bigcup\{a_i : i = 0, \ldots, l-1\} = \omega$.

We have just proved the following theorem:

Theorem 10. *If τ is the topology on ω generated by B as a base, as defined above, then $\langle\omega, \tau\rangle$ is a compact Hausdorff space with no isolated points.* □

Corollary 11. *The existence of a countable compact Hausdorff space with no isolated points is consistent with ZF.* □

This result will be improved in the next section.

Notice that in the proof above we performed several arbitrary choices, although only finitely many. However, this means that the finite subcovers cannot be obtained simultaneously for all open covers; in other words, $\langle\omega, \tau\rangle$ is not effectively compact. Of course, the results from the previous section explain why this must be true.

2.4 A Countable Connected Compact Hausdorff Space

Theorem 12. *It is consistent with ZF to have a countable, connected, compact Hausdorff space.*

Proof. We construct a model $\mathcal{N}$ of ZF where there is a connected compact Hausdorff topology on ω.

The construction of this topology could be described in the following way: Consider the Tychonoff topology on the product ${}^{\omega}\tilde{\mathbb{Q}}$, where $\tilde{\mathbb{Q}}$ stands for the additive group $\mathbb{Q}/\mathbb{Z}$ with the topology inherited from the circle $\mathbb{R}/\mathbb{Z}$. Using forcing with finite conditions, we add countably many new elements of ${}^{\omega}\tilde{\mathbb{Q}}$; this is equivalent to adding countably many Cohen reals. The enumeration of the generic elements defines a map σ from ω into ${}^{\omega}\tilde{\mathbb{Q}}$, and this map induces a topology on ω. The idea is to pass then from this generic extension to a submodel $\mathcal{N}$ where 2^{ω_0} is no longer well-orderable, but $\mathcal{N}$ contains the topology on ω (though not the embedding into ${}^{\omega}\tilde{\mathbb{Q}}$).

However, instead of using the function σ to directly induce the topology, we will choose a particular basis for the topology on ${}^{\omega}\tilde{\mathbb{Q}}$ (one formed by neighborhoods of "polyhedral" shape), and use it to define a basis B for a topology on ω; this will make easier the combinatorial topology needed for some of the arguments. We will then define the submodel $\mathcal{N}$ in such a way to make sure that $B \in \mathcal{N}$, and we will proceed to prove that the topology τ generated by B (in $\mathcal{N}$) is Hausdorff, compact, and connected.

Starting with a standard model $\mathcal{M}$ of ZFC, we use the following forcing notion to create a generic extension:

$$\mathbb{P} = \{p : p \text{ is a finite function, } \mathrm{dom}(p) \subset \omega \times \omega \text{ and } \mathrm{range}(p) \subset \tilde{\mathbb{Q}}\}.$$

This adds a collection $\{\sigma(m) : m \in \omega\}$ of new functions from ω into $\tilde{\mathbb{Q}}$. However, we can also interpret the extension as adding a collection $\{X^n_r : n \in \omega, r \in \tilde{\mathbb{Q}}\}$ of Cohen-generic subsets of ω, where $X^n_r = \{m \in \omega : (\sigma(m))_n = r\}$, for all $n \in \omega, r \in \tilde{\mathbb{Q}}$. It is easy to see that the following properties hold:

1. For each $n \in \omega$, the family $\{X^n_r : r \in \tilde{\mathbb{Q}}\}$ is a partition of ω.
2. If $n \neq n'$, then X^n_r and $X^{n'}_{r'}$ are mutually generic, for all $r, r' \in \tilde{\mathbb{Q}}$.

We define names $\dot{X}^n_r$ in such a way that $p \Vdash \check{m} \in \dot{X}^n_r$ iff $p(n, m) = r$.

We proceed to define now the sets which will eventually become the basic neighborhoods for our topology. We will think of $\tilde{\mathbb{Q}}$ as a "circle," with countably many points, so finite Cartesian products of copies of $\tilde{\mathbb{Q}}$ will be thought of as "tori." If $E \subset \omega$ is finite, the torus ${}^{E}\tilde{\mathbb{Q}}$ can be conceived also as the quotient space ${}^{E}\mathbb{Q}/{}^{E}\mathbb{Z}$. In that case, the projection $\Gamma_E : {}^{E}\mathbb{Q} \to {}^{E}\tilde{\mathbb{Q}}$ makes ${}^{E}\mathbb{Q}$ behave as a sort of covering space for ${}^{E}\tilde{\mathbb{Q}}$ (without the connectedness conditions).

We will briefly describe some notions of combinatorial topology that will be used later in the proof. All these notions are obtained using well known tools from Piecewise Linear Topology (which are usually defined in Euclidean spaces), just by taking

the intersection with ${}^{E}\mathbb{Q}$ of some of the objects in ${}^{E}\mathbb{R}$. In Appendix B we develop the elementary notions of PL Topology that are used here, and we prove some results that will be needed later in this chapter; for a complete exposition of the original objects and methods, the reader is referred to, for example, Hudson [Hud69].

Definition 13. Given $E = \{n_1, \ldots, n_k\} \subset \omega$, we call a *closed simplex* in ${}^{E}\mathbb{Q}$ the intersection with ${}^{E}\mathbb{Q}$ of a simplex in ${}^{E}\mathbb{R}$ with vertices in ${}^{E}\mathbb{Q}$. If a closed simplex is generated by exactly $k+1$ linearly independent points we call it *full dimensional.* A *closed polyhedron* in ${}^{E}\mathbb{Q}$ is the intersection with ${}^{E}\mathbb{Q}$ of a connected set in ${}^{E}\mathbb{R}$ which is the union of finitely many full dimensional closed simplices in ${}^{E}\mathbb{R}$. We will also call an *open simplex* (*open polyhedron*) the interior of a full dimensional closed simplex (closed polyhedron) with respect to ${}^{E}\mathbb{Q}$. A *simplex* (*polyhedron*) is either an open or closed simplex (polyhedron).

Remark. The topology of ${}^{E}\mathbb{Q}$ is quite different from that of ${}^{E}\mathbb{R}$; however, in our combinatorial arguments, we will use more the similarities than the differences. For example, we will think of polyhedra in ${}^{E}\mathbb{Q}$ as "connected" sets, just because they are obtained from a connected set in ${}^{E}\mathbb{R}$ (actually, the only truly connected subsets of ${}^{E}\mathbb{Q}$ are the singletons).

Given a polyhedron W in ${}^{E}\mathbb{Q}$, we use $\partial(W)$ and $\text{int}(W)$ to denote the boundary and the interior of W in the topology of ${}^{E}\mathbb{Q}$; therefore, an open polyhedron is one that is disjoint from its boundary and a closed polyhedron is one that contains it. It is clear that the interior of a simplex S of dimension less than k is empty, and its boundary is S itself. However, at some point we will use the homological boundary of simplices of dimension lower than the ambient space. In those cases, the homological boundary is defined as the intersection with ${}^{E}\mathbb{Q}$ of the homological boundary of the original simplex in ${}^{E}\mathbb{R}$ (that is, the boundary with respect to the smallest hyperplane that contains the simplex).

Other notions like subdivision and triangulation can also be easily defined in our setting, and basic results still hold. We will not use different notations for a simplicial complex and its underlying set; it should be clear from the context to which one are we referring.

Another important notion is that of a *simplicial map*. As for simplicial complexes in Euclidean spaces, a simplicial map from a complex W to a complex W' is a continuous map which sends vertices into vertices, and simplices from W linearly into (and hence onto) simplices of W'. The main point is that the vertices used always have rational coordinates; therefore, simplicial maps are piecewise linear functions with rational coefficients, and consequently map points in ${}^{E}\mathbb{Q}$ into points in ${}^{E}\mathbb{Q}$. It is clear that a simplicial homeomorphism preserves polyhedra: it is enough to take a triangulation of a polyhedron, and a refinement of the simplicial map to see that the image of the polyhedron is a finite union of full-dimensional simplices; connectedness is guaranteed by the continuity of the map. The same is true for piecewise linear homeomorphisms, since these can be transformed into simplicial maps (see Hudson [Hud69], Lemma 1.10).

Now we relativize the notions above to the tori.

Definition 14. Given $E = \{n_1, \dots, n_k\} \subset \omega$, we call an *$E$-polyhedron (in ${}^{E}\tilde{\mathbb{Q}}$)* the image of a polyhedron in ${}^{E}\mathbb{Q}$ under the projection Γ_E. An E-polyhedron W is *open* if it is disjoint from its boundary (in ${}^{E}\tilde{\mathbb{Q}}$), and it is *closed* if it contains it. If W is the image of a k-dimensional simplex S in ${}^{E}\mathbb{Q}$ whose closure is contained in an open cube of length less than 1 in ${}^{E}\mathbb{Q}$ (so that, in particular, the projection Γ_E is one-to-one on the closure of S), we call it an *E-simplex.*

Given $E = \{n_1, \dots, n_k\}$ and $\vec{r} = \langle r_1, \dots, r_k \rangle \in {}^{E}\tilde{\mathbb{Q}}$ (a "point" in the k-dimensional torus), we define

$$X_{\vec{r}} = X_{r_1}^{n_1} \cap \dots \cap X_{r_k}^{n_k}.$$

Given a set $W \subset {}^{E}\tilde{\mathbb{Q}}$, we define

$$X_W = \bigcup_{\vec{r} \in W} X_{\vec{r}}.$$

Also, we can equivalently define

$$X_{\vec{r}} = \{m : \sigma(m) \restriction E = \vec{r}\}$$

and

$$X_W = \{m : \sigma(m) \restriction E \in W\}$$

Consider the family B of all sets X_W where W is an open E-polyhedron for some $E \subset \omega$ finite. B is our intended basis for the topology on ω. To see that B is closed under finite intersections it is enough to check that if W_1 is an open E_1-polyhedron and W_2 is an open E_2-polyhedron, then $X_{W_1} \cap X_{W_2} = X_W$, where W is an open E-polyhedron with $E = (E_1 \cup E_2)$, given by:

$$W = (\pi^E_{E_1})^{-1} W_1 \cap (\pi^E_{E_2})^{-1} W_2,$$

where $\pi^E_{E_j}$ is the natural projection from ${}^{E}\tilde{\mathbb{Q}}$ onto ${}^{E_j}\tilde{\mathbb{Q}}$, for $j = 1, 2$.

Notice that if W is an E-polyhedron and $E' \supsetneq E$, then $W' = (\pi^{E'}_E)^{-1}(W)$ is an E'-polyhedron, and, moreover, it is easy to check that $X_W = X_{W'}$. However, if W is an E-simplex, W' is *not* an E'-simplex , since it "wraps around" the torus ${}^{E'}\tilde{\mathbb{Q}}$. It is still, of course, an E'-polyhedron (according to our definition). We will refer to the change from W to W' as "increasing the dimension."

The Model

Our model is defined by

$$\mathcal{N} = \mathrm{HOD}^{\mathcal{M}[G]}(\mathcal{M} \cup B \cup \{B\}),$$

where G is a $\mathbb{P}$-generic filter. Thus, B is a set in $\mathcal{N}$, and it still generates a topology on ω. This topology is clearly Hausdorff: if $m \neq m'$, then, by standard forcing arguments, there is some $n \in \omega$ such that $m \in X^n_r$ and $m' \in X^n_{r'}$, with $r \neq r'$. Taking disjoint open intervals I, I' in $\tilde{\mathbb{Q}}$ (which are $\{n\}$-polyhedra) that separate r and r', we have that X_I and $X_{I'}$ separate m and m'.

Remark. This proof can also be carried out if we define $\mathcal{N} = \mathrm{HOD}^{\mathcal{M}[G]}(B \cup \{B\})$. However, we use $\mathrm{HOD}^{\mathcal{M}[G]}(\mathcal{M} \cup B \cup \{B\})$ because it allows us to use Lemma 17 twice, saving us an awkward piece of argument.

The rest of the proof will be devoted to show that this topology is compact and connected.

Lemma 15. *Let $C \subset B$. Assume that C is closed under subsets that are members of B, and that it contains at least one element different from $\emptyset$. Also assume that C is definable in $\mathcal{M}[G]$ using only elements of $\mathcal{M} \cup \{B\}$ as parameters (not members of B). Then there exist finite sets $E_0, d \subset \omega$ such that for every open E-simplex W with $E \supset E_0$, if X_W is disjoint from d, then $X_W \in C$.*

Proof. Take $E_1 \subset \omega$, finite, such that there exists an open E_1-polyhedron W_1 with the property that $X_{W_1} \in C$ and $X_{W_1} \neq \emptyset, \omega$. It is convenient also to make sure that $|E_1| \geq 2$; this can be done by increasing the dimension, if necessary.

Take $p_1 \in \mathbb{P}$ such that $p_1 \Vdash \dot{X}_{W_1} \in \dot{C}$. Define

$$d = \{m \in \omega : \langle n, m\rangle \in \mathrm{dom}(p_1), \quad \text{for some } n \in \omega\}$$

and

$$E_2 = \{n \in \omega : \langle n, m\rangle \in \mathrm{dom}(p_1), \quad \text{for some } m \in \omega\}.$$

Take $E_0 = E_1 \cup E_2$.

Now fix any $E = \{n_1, \ldots, n_k\} \supset E_0$; we want to prove that for every open E-simplex W, if $X_W \cap d = \emptyset$ then $X_W \in C$.

Define

$$D = \{q \in \mathbb{P} : \text{if } \langle n, m\rangle \in \text{dom}(q) \text{ for some } n \in E,$$

$$\text{then } E \times \{m\} \subset \text{dom}(q)\}.$$

Clearly D is dense in $\mathbb{P}$. If $q \in D$, and $E \times \{m\} \subset \text{dom}(q)$, we define

$$\vec{r}_q(m) = \langle r_1, \ldots, r_k\rangle$$

where $q(n_j, m) = r_j$, for $j = 1, \ldots, k$. In other words, $\vec{r}_q(m)$ is the unique point $\vec{r}$ in ${}^E\tilde{\mathbb{Q}}$ such that $q \Vdash \check{m} \in \dot{X}_{\vec{r}}$. Take an extension p of p_1 such that $\text{dom}(p) = E \times d$; of course, $p \in D$.

Let now X_W be such that W is an E-simplex and X_W is disjoint from $d = \{m_1, \ldots, m_s\}$; we want to prove that $X_W \in C$.

We have that W is disjoint from the set $\{\vec{r}_p(m_1), \ldots, \vec{r}_p(m_s)\}$, otherwise X_W would contain some element of d. We make use now of the following lemma:

Lemma 16. *There exists a homeomorphism ϕ of the torus ${}^E\tilde{\mathbb{Q}}$ such that:*

1. *ϕ maps E-polyhedra onto E-polyhedra, and*
2. *ϕ maps an E-simplex $W_1 \subset W_0$ onto W, while leaving fixed each one of the points $\vec{r}_p(m_1), \ldots, \vec{r}_p(m_s)$.*

Proof. First, we choose W_1 small enough so that we can "lift" all the objects of interest into the interior of a closed $|E|$-dimensional unit cube K in the covering space ${}^E\mathbb{Q}$; in other words, we find simplices W_1', W' and points $\vec{r}_1{}', \ldots, \vec{r}_s{}'$ in K such that they are mapped by Γ_E onto W_1, W and $\vec{r}_p(m_1), \ldots, \vec{r}_p(m_s)$, respectively. Once we find a homeomorphism ϕ' of K that preserves polyhedra, maps W_1' onto W' and moves no point from $\{\vec{r}_1{}', \ldots, \vec{r}_s{}'\}$ or the boundary of K, we can define

$$\phi(\vec{r}) = \begin{cases} (\Gamma_E \circ \phi' \circ \Gamma_E^{-1})(\vec{r}), & \text{for } \vec{r} \in \Gamma_E\text{“}\,\text{int}(K) \\ \vec{r}, & \text{for } \vec{r} \in \Gamma_E\text{“}\partial(K) \end{cases}$$

To prove the existence of ϕ', we consider two cases.

For the first case, assume that there exists a closed, full-dimensional simplex S_0 contained in K (with vertices in ${}^{E}\mathbb{Q}$), such that $W_1', W' \subset \mathrm{int}(S_0)$ and $S_0 \cap \{\vec{r_1}', \ldots, \vec{r_s}'\} = \emptyset$. Then there exists another closed simplex $S_1 \subset \mathrm{int}(S_0)$ such that $W_1', W' \subset \mathrm{int}(S_1)$.

Claim. There exist piecewise linear homeomorphisms ψ_1, ψ_2 of K that map W_1', W', respectively, onto S_1, without moving any point of $K \setminus \mathrm{int}(S_0)$.

Once we have this, we can define $\phi' = \psi_2^{-1} \circ \psi_1$. Then ϕ' maps W_1 onto W_2 without moving any point of $K \setminus \mathrm{int}(S_0)$.

Proof of Claim. We follow the proof of Lemma 1.12 in Hudson [Hud69]; the central idea is the construction of a so-called *pseudo-radial projection* (*PRP*). We will show the existence of ψ_1; the existence of ψ_2 can be proved in exactly the same way.

Choose $a \in \mathrm{int}(W_1')$, and let $RP_0 \colon \partial(W_1') \to \partial(S_0)$ and $RP_1 \colon \partial(W_1') \to \partial(S_1)$ be radial projections from a. Because of the convexity of simplices, both projections are homeomorphisms; however, they fail to be piecewise linear. We will use them to find a piecewise linear homeomorphism $PRP \colon \partial(W_1') \to \partial(S_1)$.

If S is a simplex in $\partial(S_0)$ or $\partial(S_1)$, we can form a full-dimensional simplex S^* by taking the convex hull of $S \cup \{a\}$. Consider then a subdivision Δ of $\partial(W_1')$ which contains subdivisions of the polyhedra

$$RP_j^{-1}(S) = \partial(W_1') \cap S^*$$

for each $S \in \partial(S_j)$, for $j = 0, 1$.

Now let T be a simplex in Δ. We have that RP_1"T is a simplex contained in a face of $\partial(S_0)$. Define $PRP \colon \Delta \to \partial(S_0)$ by letting $PRP(b) = RP_1(b)$ for all vertices b of Δ, and then extending linearly. Then PRP is a well-defined map, and it is a piecewise linear homeomorphism from $\Delta = \partial(W_1')$ to $\partial(S_1)$.

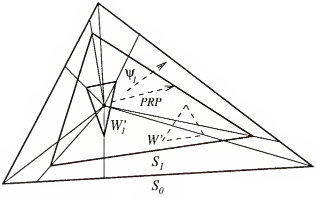

Figure 2.1: The map ψ_1, in the two-dimensional case.

Finally, to define ψ_1, we consider a subdivision Δ' of S_0 that contains $\{a\}$, Δ and S_1, but adding no new vertices. Once ψ_1 is defined for all vertices, it extends linearly to all the simplices to become a piecewise linear continuous function (which will be a homeomorphism because of the way it is defined). If b is a vertex in Δ', then

$$\psi_1(b) = \begin{cases} a, & \text{if } b = a \\ PRP(b), & \text{if } b \in \Delta = \partial(W_1') \\ b, & \text{if } b \in \partial(S_0) \end{cases}$$

and if $b \in \partial(S_1)$, then we define $\psi_1(b)$ as follows: We know that b is in the line segment between $PRP^{-1}(b) \in \Delta$ and the radial projection c of b into $\partial(S_0)$. Then, since $\psi_1(PRP^{-1}(b))$ is defined, and we know that $\psi_1(c)$ will eventually be defined as c, we define $\psi(b)$ linearly to a point in the segment between b and c. This allows us to complete the definition of ψ_1, and it finishes the proof of the claim. $\square$

For the general case, assume that there is no such simplex S. Using the fact that the dimension $|E| \geq 2$, we can find a polygonal path γ connecting W_1' and W', while simultaneously avoiding the points $\vec{r}_1{}', \ldots, \vec{r}_s{}'$. By compactness arguments (in ${}^E\mathbb{R}$), there exists a finite collection $\{S_1, S_2, \ldots, S_t\}$ of open simplices such that the following hold:

1. $\gamma \subset \bigcup\{S_1, S_2, \ldots, S_t\}$,

2. $\{\vec{r}_1', \ldots, \vec{r}_s'\} \cap \bigcup\{S_1, S_2, \ldots, S_t\} = \emptyset$,
3. $W_1' \subset S_1$ and $W' \subset S_t$, and
4. $S_i \cap S_{i+1} \neq \emptyset$, for $i = 1, \ldots, t-1$.

By choosing simplices $W_2', \ldots, W_{t-1}', W_t' = W'$ such that $W_{i+1}' \subset S_i \cap S_{i+1}$ for $i = 1, \ldots, t-1$, we can repeat the argument from the first case above to find polyhedron preserving homeomorphisms $\phi_1', \ldots, \phi_t'$ such that, for each $i = 0, \ldots, t-1$,

1. ϕ_i' maps W_i' onto W_{i+1}'.
2. ϕ_i' moves no points outside of S_i.

We then obtain our desired homeomorphism by choosing $\phi' = \phi_t' \circ \cdots \circ \phi_0'$. □

The homeomorphism ϕ we have just found induces an order automorphism ϕ^* on $D \subset \mathbb{P}$, defined by

$$(\phi^* q)(n_j, m) = (\phi(\vec{r}_q(m)))_j$$

for any m such that $E \times \{m\} \subset \mathrm{dom}(q)$ and $j = 1, \ldots, k$, and $(\phi^* q)(n, m) = q(n, m)$ if $\langle n, m\rangle \in \mathrm{dom}(q)$ but $n \notin E$.

Now, since D is dense in $\mathbb{P}$, we can consider D as an equivalent notion of forcing, and in fact, $\mathcal{M}[G \cap D] = \mathcal{M}[G]$.

The sets of the form X_Z, where Z is an E-polyhedron, have a natural D-name $\dot{X}_Z$ given by $\dot{X}_Z(\check{m}) = \{q \in D : E \times \{m\}\, \mathrm{dom}(q) \text{ and } \vec{r}_q(m) \in Z\}$. Since $W_1 \subset W_0$, then $\Vdash \dot{X}_{W_1} \subset \dot{X}_{W_0}$. Also, the name $\dot{C}$ can be chosen, without loss of generality, so that it is a D-name and $\Vdash \forall X, Y \in \dot{B}(Y \in \dot{C} \wedge X \subset Y \to X \in \dot{C})$. Therefore, we have that $p \Vdash \dot{X}_{W_0} \in \dot{C}$.

Then, by the usual arguments about automorphisms of the forcing notion, (as can be found, for example, in Jech [Jec78]), ϕ^* can be extended to the class of D-names, and we have that

$$\phi^* p \Vdash \phi^* \dot{X}_{W_1} \in \phi^* \dot{C}$$

It can be easily verified that $\phi^*\dot{X}_{W_1} = \dot{X}_{\phi``W_1}$ and that $\phi^* p = p$, so we have

$$p \Vdash \dot{X}_{\phi``W_1} \in \phi^*\dot{C}.$$

Now, since ϕ^* leaves invariant the canonical names of elements of $\mathcal{M}$ and the natural name $\dot{B}$, it follows that the interpretation of $\phi^*\dot{C}$ under $G \cap D$ is C, because it satisfies the same defining formula with the same parameters. Therefore,

$$p \Vdash \dot{X}_W \in \dot{C}'$$

where $\dot{C}'$ is a name for C. Since $p \in G$, we conclude that $X_W \in C$, finishing the proof of Lemma 15. $\square$

This lemma allows us to prove the finite subcover property for the special case when a given open cover C for ω is definable from B and elements of $\mathcal{M}$ (we can assume without loss of generality that $C \subset B$ and that it is closed under subsets in B):

Lemma 17. *Let $\mathcal{M}, \mathcal{N}, B$ be defined as before, and let $C \subset B$ be a cover for ω, closed under subsets in B. Assume that C is definable from elements of the ground model and B (using no elements of B). Then there exists a finite subcover $\bar{C} \subset C$ for ω.*

Proof. Take E_0, d as in the conclusion of Lemma 15, and take $|d|$ sets $X_{W_1}, \ldots, X_{W_s}$ from C, where each W_j is an E_j-polyhedron, to cover each $m \in d$. By increasing dimension, if necessary, we can consider each W_j to be an E-polyhedron, where $E = E_0 \cup E_1 \cup \cdots \cup E_s$. Since C is closed under subsets in B, we can then find E-simplices $S_j \subset W_j, j = 1, \ldots, s$ such that $\bigcup_{j=1,\ldots,s} X_{S_j} \supset d$, but at the same time, the closures of $S_j, S_{j'}$ are disjoint if $j \neq j'$.

Now triangulate the torus ${}^{E}\tilde{\mathbb{Q}}$ in such a way that $S_j, j = 1, \ldots, s$ form part of a triangulation of the torus in E-simplices. Let $S'_{s+1}, \ldots, S'_r$ be the rest of the E-simplices of the triangulation, taken as open E-simplices. We enlarge slightly each S'_j to obtain new open E-simplices $S_{s+1}, \ldots, S_r$ such that

$$S_1 \cup \cdots \cup S_s \cup S_{s+1} \cup \cdots \cup S_r = {}^{E}\tilde{\mathbb{Q}}.$$

but taking care that no one of the points in d enters any of the $S_{s+1}, \ldots, S_r$. Then, by the properties of E_0 and d, the sets $X_{S_j}, j = s+1, \ldots, r$ are actually members of C. Therefore, $\bar{C} = \{X_{S_j} : j = 1, \ldots, r\}$ is a finite subcover of C. $\square$

For the general case, we will proceed as follows: Let C be an open cover for ω which is definable from elements of $\mathcal{M}$, B, and the parameters $X_{W_1}, \ldots, X_{W_u}$. Choose E_0 large enough so that $W_1, \ldots, W_u$ are all E_0-polyhedra. Triangulate each W_j, and then produce a triangulation $\{S_i : i = 1, \ldots, v\}$ of ${}^{E_0}\tilde{\mathbb{Q}}$ in such a way that all the triangulations of the sets W_j are contained in it (refining each one of these if necessary). It is enough then to find a finite subcover for each set $X_{S_i}, i = 1, \ldots, v$, since the union of these subcovers is a finite subcover for ω. We will proceed by induction on the dimension of the simplexes, and use the fact that each X_{S_i} is either disjoint or contained in each of the parameters $X_{W_1}, \ldots, X_{W_u}$.

For dimension 0, S_i is a singleton $\{\vec{r}_0\}$, with $\vec{r}_0 \in {}^{E_0}\tilde{\mathbb{Q}}$. We will base our proof on the fact that, in this case, X_{S_i} resembles the whole space. In order to be able to apply Lemma 17, we will factor the extension from $\mathcal{M}$ to $\mathcal{M}[G]$ in two parts, so that the parameters $X_{W_j}, j = 1, \ldots, u$ can be considered as elements of the first extension, and then consider this first extension as the ground model for the second extension.

It is easy to see that $\mathbb{P}$ is isomorphic to the product $\mathbb{P}_0 \times \mathbb{P}_1$, where

$$\mathbb{P}_0 = \{p_0 : p_0 \text{ is a finite function, } \mathrm{dom}(p_0) \subset E_0 \times \omega \text{ and } \mathrm{range}(p_0) \subset \tilde{\mathbb{Q}}\}$$

and

$$\mathbb{P}_1 = \{p_1 : p_1 \text{ is a finite function,}$$
$$\operatorname{dom}(p_1) \subset (\omega \setminus E_0) \times \omega \text{ and } \operatorname{range}(p_1) \subset \tilde{\mathbb{Q}}\}.$$

If we define

$$G_0 = \{p_0 \in \mathbb{P}_0 : \langle p_0, p_1 \rangle \in G \text{ for some } p_1 \in \mathbb{P}_1\}$$

and

$$G_1 = \{p_1 \in \mathbb{P}_1 : \langle p_0, p_1 \rangle \in G \text{ for some } p_0 \in \mathbb{P}_0\},$$

then, by well known results about generic extensions (see, for example, Jech [Jec78], Lemma 20.1) we have that G_0 is generic over M and G_1 is generic over $\mathcal{M}[G_0]$; furthermore, $\mathcal{M}[G] = \mathcal{M}[G_0][G_1]$.

The union of the generic set G_0 is a function $\sigma_0 \colon \omega \to {}^{E_0}\tilde{\mathbb{Q}}$ to $\mathcal{M}$, while the union of G_1 is a function $\sigma_1 \colon \omega \to {}^{(\omega \setminus E_0)}\tilde{\mathbb{Q}}$ to $\mathcal{M}[G_0]$; from σ_0 and σ_1 we can recover the function $\sigma \colon \omega \to {}^{\omega}\tilde{\mathbb{Q}}$ in $\mathcal{M}[G]$ in the obvious way. We have that, for any $\vec{r} \in {}^{E_0}\tilde{\mathbb{Q}}$, $\sigma_0^{-1}(\vec{r}) = X_{\vec{r}}$ (in $\mathcal{M}[G]$). Since $\sigma_0 \in \mathcal{M}[G_0]$, that means that for every $\vec{r} \in {}^{E_0}\tilde{\mathbb{Q}}$, $X_{\vec{r}} \in \mathcal{M}[G_0]$. In consequence, all sets $X_{W_j}, j = 1, \ldots, u$ are also in $\mathcal{M}[G_0]$, since each E_0-polyhedron is an element of $\mathcal{M}$.

Now, in $\mathcal{M}[G_0]$, we can identify $\mathbb{P}_1$ with $\mathbb{P}$ using a bijection $\xi \colon \omega \to \omega \setminus E_0$ in $\mathcal{M}$; ξ extends naturally to an isomorphism $\xi_* \colon \mathbb{P}_1 \to \mathbb{P}$ given by

$$\xi_* \colon p \mapsto p \circ \xi.$$

Since $\xi_* \in \mathcal{M}$, $G_1' = \xi_*``G_1$ is $\mathbb{P}$-generic over $\mathcal{M}[G_0]$, and $\mathcal{M}[G_0][G_1'] = \mathcal{M}[G]$. The union G_1' is a function $\sigma' \colon \omega \to {}^{\omega}\tilde{\mathbb{Q}}$.

We can also define a bijection $\xi^* \colon X_{\vec{r}} \to \omega$, given by $\xi^* \colon m \mapsto m'$, where m' is the unique number such that

$$\sigma'(m') = \sigma_1(m) \circ \xi.$$

Claim: $X \subset X_{\vec{r}}$ is the intersection of a basic neighborhood with $X_{\vec{r}}$ if and only if $\xi^*``X$ is a basic neighborhood of $\langle \omega, \tau \rangle$ as computed in $(\mathcal{M}[G_0])[G_1']$.

To prove the claim, let $X = X_{\vec{r}} \cap X_W$, where W is an open E_W-polyhedron and we can assume (by increasing the dimension, if necessary) that $E_W \supset E_0$. Let $\bar{W}$ be a full-dimensional open polyhedron in ${}^{E_W}\mathbb{Q}$ such that $\Gamma_{E_W}``\bar{W} = W$. We have that the intersection $\bar{W} \cap {}^{E_W \setminus E_0}\mathbb{Q}$ is a full-dimensional, open polyhedron in the hyperplane ${}^{E_W \setminus E_0}\mathbb{Q}$. Since the map

$$\bar{\xi}^* \colon \langle t_j, j \in E_W \setminus E_0 \rangle \mapsto \langle t_j' = t_{\xi(j)}, j \in \xi^{-1}(E_W \setminus E_0) \rangle,$$
$$\forall \langle t_j, j \in E_W \setminus E_0 \rangle \in {}^{E_W \setminus E_0}\mathbb{Q}$$

is just a relabeling of the axes, it is a linear isometry, and consequently the image of $\bar{W} \cap {}^{E_W \setminus E_0}\mathbb{Q}$ is a full-dimensional, open polyhedron in ${}^{\xi^{-1}(E_W \setminus E_0)}\mathbb{Q}$. The projection under $\Gamma_{E_W \setminus E_0}$ of that image is an open $\xi^{-1}(E_W \setminus E_0)$-polyhedron W'; we have that

$$\begin{aligned} \xi^*``X =& \{m' : \sigma'(m') = \sigma_1(m) \circ \xi \text{ and } m \in X\} \\ =& \{m' : \sigma'(m') = \sigma_1(m) \circ \xi \text{ and } \sigma_1(m) \restriction E_W \in W \text{ and } \sigma_1(m) \restriction E_0 = \vec{r}\} \\ =& \{m' : \sigma'(m') \restriction \xi^{-1}(E_W \setminus E_0) \in W'\}, \end{aligned}$$

and this is just $X_{W'}$ as computed in $(\mathcal{M}[G_0])[G_1']$. Therefore, it is a basic neighborhood of $\langle \omega, \tau \rangle$ as computed in $(\mathcal{M}[G_0])[G_1']$.

For the other direction of the claim, assume that $\xi^*``X$ is a basic neigborhood $X_{W'}$ of $\langle \omega, \tau \rangle$ as computed in $(\mathcal{M}[G_0])[G_1']$, where $W'$ is an open $E_{W'}$-polyhedron. Reversing the process above, we can obtain an open, full-dimensional polyhedron $W''$ in the hyperplane ${}^{(\xi``E_{W'})}\mathbb{Q}$ of the space ${}^{E_0 \cup \xi``E_{W'}}\mathbb{Q}$. From $W'$ we can obtain a full-dimensional polyhedron $\bar{W}$ in ${}^{E_0 \cup \xi``E_{W'}}\mathbb{Q}$ by taking

$$\bar{W} = \{\langle t_j : j \in E_0 \cup \xi``E_{W'} \rangle : \langle t_j \rangle \restriction E_{W'} \in W'' \text{ and } 0 \leq t_j \leq 1 \forall j \in E_0.$$

Then, $\Gamma_{E_0 \cup \xi``E_{W'}}``\bar{W}$ is an open $E_0 \cup \xi``E_{W'}$-polyhedron and $X_W \cap X_{\vec{r}}$ (computed in $\mathcal{M}[G]$) is equal to $X_{W'}$ (computed in $(\mathcal{M}[G_0])[G_1'])$. This finishes the proof of the claim.

We use the claim to prove that the cover C of $X_{S_i} = X_{\vec{r}}$ has a finite subcover. The set

$$\bar{C} = \{\xi^* ``X : X \in C\}$$

is a cover for $\langle \omega, \tau \rangle$ as computed in $(\mathcal{M}[G_0])[G_1']$. Since C is definable from $W_1, \ldots, W_u$, and these are all elements of $\mathcal{M}[G_0]$, we can apply Lemma 17 to obtain a finite subcover $\bar{C}' \subset \bar{C}$ for ω. Then, the set $\{X \in C : \xi^* ``X \in \bar{C}'\}$ is a finite subcover for X_{S_i}.

This finishes the initial case.

Now, for the inductive step, the inductive hypothesis allows us to find a finite subcover for each one of the (finitely many) simplices that form the homological boundary of the given simplex S_i. All that remains is to cover the part of the interior of S_i not covered yet by the covers of the homological boundary.

For this argument we adapt the techniques from Lemmas 15 and 17.

Let T be the complex obtained as the union of all simplices that contain S_i (this can be called $\widetilde{\text{star}}(S_i; K)$, where K is the triangulation of the space; see Appendix B for the definition of $\widetilde{\text{star}}$ in the Euclidean case). We have three cases:

Case 1: $1 < \dim(S_i) = |E_0|$. In this case, T=S_i.

The proofs of Lemmas 15 and 17 can be adapted to work inside S_i. That is, we can find finite subsets E_1, d of ω such that $E_1 \supset E_0$, and for every open E-simplex W with $E \supset E_1$, if $W \supset S_i$ and $X_W \cap d = \emptyset$, then $X_W \in C' = \{X \cap X_{S_i} : X \in C\}$. The key lies in the fact that all the piecewise linear homeomorphisms needed in the argument can be taken such that the complement of S_i and the boundary of S_i are fixed. This way, all these homeomorphisms leave invariant each one of the parameters $X_{W_1}, \ldots, X_{W_s}$, and we can avoid the assumption that all the parameters are in the

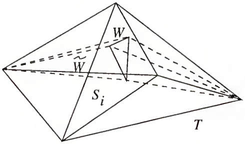

Figure 2.2: The set $\widetilde{W}$ (three-dimensional view)

ground model. After repeating the proof of Lemma 17 in this setting, the final result is that we find a finite cover for the interior of S_i.

Case 2: $1 < \dim(S_i) < |E_0|$.

The whole argument from Case 1 could be carried out inside the hyperplane that contains S_i (notice that only elements in the interior of S_i, with respect to the hyperplane, would be moved), if we could extend the piecewise linear homeomorphisms in the hyperplane to the entire space. To show how this extension is achieved, we lift the complex T up to a cube of side 1 in ${}^{E_0}\mathbb{Q}$. The objects will be called T' and S'_i; we have that $T' = \widetilde{\operatorname{star}}(S'_i; K')$, where K' is the induced triangulation of the cube. Using Lemma 2 of Appendix B, we obtain that all the needed piecewise linear homeomorphisms can be extended to the whole cube, and then projected to ${}^{E_0}\tilde{\mathbb{Q}}$.

If W is an open polyhedron contained in S_i (open with respect to the hyperplane determined by S_i), we define

$$\widetilde{W} = \Gamma_{E_0}\text{``}\{\alpha\vec{r}_0 + \beta\vec{r}_1 : \alpha + \beta = 1, \alpha, \beta \geq 0, \vec{r}_0 \in W',$$
$$\text{and } \vec{r}_1 \text{ is in a subsimple of } T \text{ disjoint from } S'_i\}$$

where W' is a polyhedron contained in S'_i such that $\Gamma_{E_0}\text{``}W' = W$. We define now a subcover of C which is easier to handle:

$$\widetilde{C} = \{X_{W \cap (\widetilde{W \cap S_i})} : X_W \in C\}.$$

Clearly, $\widetilde{C}$ is closed under subsets of the form $X_{W\cap(\widetilde{W\cap S_i})}$. Using piecewise linear homeomorphisms like $\Gamma_{E_0} \circ \bar{\phi}\Gamma_{E_0}^{-1}$, where $\bar{\phi}$ is defined as above, we can repeat the argument for Case 1, and find a finite subcover of $\widetilde{C}$ for the interior of S_i.

Case 3: $\dim S_i = 1$.

The problem in this case is the same, whether $|E_0| = 1$ or not. It arises if the set d of numbers involved in the forcing condition used is not empty, because in this case the proof of Lemma 16 fails: there is no path connecting simplices on opposite sides of a point $\vec{r}$ such that an element of d appears in $X_{\vec{r}}$.

To avoid this problems, we increase the dimension to 2. Here we can apply the techniques from either one of the cases above to find a finite cover for the interior of X_{S_i}. Notice that this argument is not circular: the inductive hypothesis for the 2-dimensional case was used only to cover the homological boundaries of the simplex; here, we are using the techniques described above to cover the interior.

This finishes the proof of the existence of a finite subcover for the general case, proving that $\langle \omega, \tau \rangle$ is compact.

Finally, to prove connectedness, we need a last lemma.

Lemma 18. *Let A be a non-empty open set in ω. Then there exists $E \subset \omega$ finite, and E-polyhedra $W_1, \ldots, W_u$ such that*

$$X_{W_1} \cup \cdots \cup X_{W_u} \subset A$$

and

$$A \setminus X_{W_1} \cup \cdots \cup X_{W_u} = X_{S_1} \cup \cdots \cup X_{S_v}$$

where $S_1, \ldots, S_v$ are simplices in ${}^E\tilde{\mathbb{Q}}$ of dimension strictly lower than $|E|$.

Proof. Let $C = \{X \in B : X \subset A\}$. C is closed under subsets in B; let $X_{U_1}, \ldots, X_{U_t}$ be the parameters used in the definition of C, and take E large enough for $U_1, \ldots, U_t$ to be E-polyhedra. Produce a triangulation of of ${}^E\tilde{\mathbb{Q}}$ that also triangulates every U_j,

as in the proof of compactness above. Let $\{W_1, \ldots, W_u\}$ be the set of simplices in the triangulation that have non-empty intersection with A. There is an element of C contained in each one of $\{W_1, \ldots, W_u\}$. By arguments as in the proof of Lemma 15, every element inside each X_{W_j} can be covered by an element of C, except may be a finite set. Therefore, A consists of a union of the basic neighborhoods $X_{W_1}, \ldots, X_{W_u}$, plus, may be, finitely many sets of the form X_S, where S is the $(|E| - l)$-dimensional simplex shared by $l+1$ members of $\{W_1, \ldots, W_u\}$, without the homological boundary. To this, we might substract a finite number of interior points left out above (in sets like d from Lemma 15). $\square$

Now we are able to prove connectedness. Given an open set A, we just saw that it is, essentially, a finite union of basic neighborhoods. If we take the closure of A, it is the union of the closures of the finitely many open sets that conform it. But since the closure of X_W, for W an open E-polyhedron, is $X_{\bar{W}}$, where $\bar{W}$ is the closure of W in ${}^E\tilde{\mathbb{Q}}$, the only case when the closure of A is equal to A is when A is the whole space. This means that ω is the only non-empty clopen set, that is, that the topology on ω is connected. $\square$

CHAPTER 3
JÓNSSON CARDINALS AND SINGULARITY

3.1 Introduction

Mitchell [Mit99] proved that if there is no inner model with a Woodin cardinal and the Steel core model K exists, then every Jónsson cardinal (in V) is a Ramsey cardinal in K; furthermore, if κ is a δ-Jónsson cardinal in V and δ is regular in V, then κ is a δ-Erdős cardinal in K. The same results are obtained in the absence of the Steel core model K if some extra assumptions are made.

In the same paper it is asked how can these results be generalized to include singular cardinals. This chapter deals with one of the possible cases, namely, when δ is singular. However, in order to simplify the arguments, we will assume stronger hypotheses that allow us to use the theory of the core model for sequences of measures. This way we avoid the need to use iteration trees while at the same time we retain the essential features of the arguments. It seems plausible that the results obtained here can also be obtained using hypotheses similar to those of Mitchell [Mit99], especially since the structure of the proof here follows closely that of the proof in that paper. It must be mentioned that the proof presented here is just an extension of the proof in Mitchell [Mit99], and that most of the machinery used comes from that and other papers by Mitchell. Also, we are indebted to him for his many suggestions and guidance in the development of this proof.

3.2 Basic Concepts

Definition 1. If $\delta \leq \kappa$ are cardinals, then κ is said to be *δ-Jónsson* if for each first order structure $\mathcal{A}$ in a countable language with universe κ there exists an elementary substructure $\mathcal{A}' \prec \mathcal{A}$ with universe $A' \neq \kappa$ such that the order type of A' is δ.

Notice that according to this definition, κ is κ-Jónsson if and only if it is a Jónsson cardinal.

Definition 2. If $\delta \leq \kappa$ are cardinals, then κ is said to be *δ-Erdős* if for every structure $\mathcal{A}$ in a countable language with universe κ, and for every closed unbounded subset C of κ, there is a set $D \subset C$ of order type δ which is a normal set of indiscernibles for $\mathcal{A}$.

In this setting, a *normal set of indiscernibles* is a set D such that for every n-ary function f which is definable in $\mathcal{A}$ without parameters, either $f(\vec{d}) \geq d_0$ for every $\vec{d} = \langle d_0, \ldots, d_{n-1} \rangle \in [D]^n$, or else the value of $f(\vec{d})$ is constant for $\vec{d} \in [D]^n$.

Proposition 3. *Every δ-Erdős cardinal κ such that* $\mathrm{cof}(\kappa) > \omega$ *is δ-Jónsson.*

Proof. Let κ be δ-Erdős and let $\mathcal{A}$ be a first order structure over a countable language, with universe κ.

Assume without loss of generality that $\mathcal{A}$ has Skolem functions. The set C of ordinals $\nu < \kappa$ which are closed under the Skolem functions is closed unbounded, since $\mathrm{cof}(\kappa) > \omega$. Let $D \subset C$ be a normal set of indiscernibles for $\mathcal{A}$.

Since $D \subset C$, the Skolem hull $\mathcal{H}^{\mathcal{A}}(D)$ is contained in $\sup(D)$. The fact that D is a normal set of indiscernibles implies that the set

$$F(D) = \bigcup \{ f``D : f \text{ is a Skolem function of } \mathcal{A} \}$$

satisfies that for all $\nu < \sup(D)$, $|F(D) \cap \nu| \leq |D \cap \nu| \cdot \aleph_0 < \delta$. Similarly, if we define

$$F^0(D) = F(D)$$

and

$$F^{n+1}(D) = F(F^n(D)) = \bigcup \{ f``F^n(D) : f \text{ is a Skolem function of } \mathcal{A} \},$$

then we have that for each $n \in \omega$ and for each $\nu < \sup(D)$, the set $F^n(D)$ satisfies that $|F^n(D) \cap \nu| \leq |D \cap \nu| \cdot \aleph_0 < \delta$. Therefore the set $\mathcal{H}^{\mathcal{A}}(D) = \bigcup_{n \in \omega} F^n(D)$ also

satisfies that $|\mathcal{H}^{\mathcal{A}}(D) \cap \nu| \leq |D \cap \nu| \cdot \aleph_0 < \delta$ for each $\nu < \sup(D)$, and we can conclude that $\mathcal{H}^{\mathcal{A}}(D)$ has order type δ. Since $\mathcal{H}^{\mathcal{A}}(D) \prec \mathcal{A}$, we have proved that κ is δ-Jónsson. □

3.3 The Theorem

Theorem 4. *Assume that for all cardinals η, $o(\eta) < \eta^{++}$, and let $K = L[\mathcal{U}]$ be the core model for sequences of measures. Let $\delta < \kappa$ be cardinals such that $\omega < \mathrm{cof}(\delta) < \delta < \mathrm{cof}(\kappa)$, and assume that κ is δ-Jónsson. Then, at least one of the following is true:*

1. *κ is δ-Erdős.*
2. *The set L of cardinals λ with $\mathrm{cof}(\lambda) = \mathrm{cof}(\delta)$ which are limits of measurable cardinals in K is stationary in κ.*

Proof. Let κ be a δ-Jónsson cardinal. Let $C \in K$ be a club set of κ, and $\mathcal{A}$ a structure on a countable language with universe κ.

Proposition 5. *There is a set X, with $\{\delta, \kappa, C, \mathcal{A}\} \subset X$ and $(X, \mathcal{U}) \prec (H_\lambda, \mathcal{U})$, such that $\delta \not\subset X$ but $\mathrm{o.t.}(X \cap \kappa) = \delta$.*

Proof. First, let $(X^*, \mathcal{U})$ be an elementary substructure of $(H_\lambda, \mathcal{U})$ such that $\{\delta, \kappa, C, \mathcal{A}\} \cup \kappa \subset X^*$ and $|X^*| = \kappa$; such set can be found by taking the Skolem hull of $\kappa+1 \cup \{C, \mathcal{A}\}$ in $(H_\lambda, \mathcal{U})$.

Now let $f\colon \kappa \cong X^*$ be a bijection, and code the structure $(X^*, \mathcal{U})$ into a structure with universe κ. We extend that structure to a structure $\mathcal{B}$ that has function symbols interpreted by $f^{-1} \restriction \kappa$ and $f \cap (\kappa \times \kappa)$. Thus, if $Z \prec \mathcal{B}$, then $f^{-1}Z \subset Z$, and that means that $f``Z$ (which is the universe of an elementary substructure of $(X^*, \mathcal{U})$), satisfies $Z \subset f``Z$. So $\kappa \cap f``Z \supset Z$. But also, if $\alpha \in \kappa \cap f``Z$, then $\alpha \in Z$, since Z is closed under $f \cap (\kappa \times \kappa)$. Therefore, $\kappa \cap f``Z = Z$, for all $Z \prec \mathcal{B}$.

If $\delta = \kappa$, then κ is Jónsson, and $\mathcal{B}$ has an elementary substructure Z such that $|Z| = \kappa$ but $Z \neq \kappa$. If $\delta < \kappa$, then $\mathcal{B}$ has an elementary substructure Z such that

o.t.$(X) = \delta$. Since $\delta \in f``Z \cap \kappa = Z$, we cannot have $\delta \subset Z$; since in that case we would have o.t.$(Z) > \delta$. If we take $X = f``Z$, then $(X,\mathcal{U}) \prec (H_\lambda,\mathcal{U})$, and $X \cap \kappa = Z$, so o.t.$(X \cap \kappa) = \delta$. □

We use the previous proposition to obtain X with the prescribed properties. Let N be the transitive collapse of X, with collapsing map

$$\pi : N \cong X.$$

Then, $\mathrm{crit}(\pi) = \min(\kappa \setminus X) < \delta$. Also, notice that κ is collapsed to δ, so $\pi(\delta) = \kappa$.

Set $\overline{W} = \pi^{-1}(X \cap K)$. We will run a comparison between $\overline{W}$ and K, with iterations $\langle \mathcal{M}_\nu \rangle$ on K and $\langle \mathcal{N}_\nu \rangle$ on $\overline{W}$; however, this comparison will have non–standard characteristics. The following diagram shows the iterations involved:

$$\begin{array}{lccc}
K = \mathcal{M}_0 : & \mathcal{M}_\nu & \overset{i_{\nu,\nu'}}{\rightsquigarrow} & \mathcal{M}_{\nu'} \\
 & & & \\
\overline{W} = \mathcal{N}_0 : & \mathcal{N}_\nu & \overset{j_{\nu,\nu'}}{\rightsquigarrow} & \mathcal{N}_{\nu'} \\
\big\downarrow \pi & \pi_\nu \big\downarrow & & \big\downarrow \pi_{\nu'} \\
K = \widetilde{\mathcal{N}}_0 : & \widetilde{\mathcal{N}}_\nu & \overset{\tilde{j}_{\nu,\nu'}}{\rightsquigarrow} & \widetilde{\mathcal{N}}_{\nu'}
\end{array}$$

The wavy arrows express the fact that the corresponding embeddings might be undefined for a particular pair ν, ν'; this happens when there is a drop (that is, a stage γ such that the embedding from from the γ-th model to the $\gamma + 1$-st model is not defined) between the ν-th and the ν'-th stages of the iteration.

The iterations $\langle \mathcal{N}_\nu \rangle$ and $\langle \widetilde{\mathcal{N}}_\nu \rangle$ are standard, but the iteration $\langle \mathcal{M}_\nu \rangle$ is specially defined. We will define all of them simultaneously by induction as a comparison process.

3.4 Construction of the Iterations

Let $\mathcal{M}_0 = K$, $\mathcal{N}_0 = \overline{W}$, and $\widetilde{\mathcal{N}}_0 = K$; also, let $\pi_0 = \pi\colon \mathcal{N}_0 \to \widetilde{\mathcal{N}}_0$. Suppose that during the course of the recursion we have already defined initial segments $\langle \mathcal{M}_\nu \rangle_{\nu<\phi}$, $\langle \mathcal{N}_\nu \rangle_{\nu<\theta}$, and $\langle \widetilde{\mathcal{N}}_\nu \rangle_{\nu<\theta}$, as well as elementary embeddings $\pi_\nu\colon \mathcal{N}_\nu \to \widetilde{\mathcal{N}}_\nu$ for all $\nu < \theta$. We also assume that there exist final segments of each iteration for which the elementary embeddings $i_{\nu,\nu'}\colon \mathcal{M}_\nu \to \mathcal{M}_{\nu'}$, for $\nu < \nu' < \phi$, $j_{\nu,\nu'}\colon \mathcal{N}_\nu \to \mathcal{N}_{\nu'}$, for $\nu < \nu' < \theta$, and $\tilde{\jmath}_{\nu,\nu'}\colon \widetilde{\mathcal{N}}_\nu \to \widetilde{\mathcal{N}}_{\nu'}$, for $\nu < \nu' < \theta$, are all defined and form directed systems, while the embeddings π_ν, $pi_{\nu'}$ commute with the embeddings $j_{\nu,\nu'}$ and $\tilde{\jmath}_{\nu,\nu'}$. This assumption will be shown to hold during the recursive definition by checking that there are only finitely many drops before each stage.

At the next stage we extend one or both of the iterations, depending on which of the following three cases occurs.

Case 1. At least one of ϕ or θ is a limit ordinal.

If θ is a limit ordinal, then let $\mathcal{N}_\theta$, $\widetilde{\mathcal{N}}_\theta$, and π_θ be the transitive collapse of the direct limits. The direct limit of $\langle \widetilde{\mathcal{N}}_\nu \rangle$, $\nu < \theta$, is well founded because the iteration $\langle \widetilde{\mathcal{N}}_\nu \rangle$ is standard and K is iterable. Since the embedding π'_θ, defined as the direct limit of the embeddings $\pi_\nu, \nu < \theta$, maps the direct limit of $\langle \mathcal{N}_\nu \rangle$ into the direct limit of $\langle \widetilde{\mathcal{N}}_\nu \rangle$ while preserving membership, we obtain that the direct limit of $\langle \mathcal{N}_\nu \rangle$ is also well founded.

If ϕ is a limit ordinal, we cannot conclude immediately that the next iteration works, because it is a non-standard iteration. We will use the following lemma; the proof is similar to that of Lemma 3.2 in Mitchell [Mit99].

Lemma 6. *Assume that γ is a limit ordinal, and that the models $\mathcal{M}_\nu$, $\nu < \gamma$, are obtained using the non-standard iteration $\langle \mathcal{M}_\nu \rangle$, with embeddings $i_{\nu,\nu'}$ defined for all $\nu, \nu' < \gamma$ such that ν, ν' are larger than the last level where there was a drop. Then the direct limit of the directed system $\langle \mathcal{M}_\nu, i_{\nu',\nu''} : \nu, \nu'\nu'' \rangle$ is a well-founded model.*

As usual, we define $\mathcal{M}_\gamma$ as the transitive collapse of the direct limit of $\langle \mathcal{M}_\nu, i_{\nu',\nu''} : \nu\nu'\nu''\rangle$.

This concludes Case 1. In the remaining cases both ϕ and θ are successor ordinals, say, $\phi = \gamma + 1$ and $\theta = \gamma' + 1$. Let α be the largest ordinal such that $\mathcal{M}_\gamma$ and $\mathcal{N}_{\gamma'}$ agree up to α. Also let $\tilde{\alpha} = \sup \pi_{\gamma'}``\alpha$, and

$$\mathcal{R} = \mathrm{ult}(\mathcal{M}_\gamma, \pi_{\gamma'}, \tilde{\alpha}) = \{[a, f] : a \in [\tilde{\alpha}]^{<\omega} \text{ and } f \in \mathcal{M}_\gamma\}$$

(the definition of this kind of ultrapower is given in Mitchell [Mit99]).

Case 2. $\mathcal{M}_\gamma$ is a proper class and $\mathcal{R}$ is not iterable.

This is the case that makes the iteration $\langle \mathcal{M}_\nu \rangle$ non-standard. We have the following lemma, which can be proved in a similar way as Lemma 3.3 of Mitchell [Mit99].

Lemma 7. *If $\mathcal{R} = \mathrm{ult}(\mathcal{M}_\gamma, \pi_{\gamma'}, \tilde{\alpha})$ is not iterable, then there is a substructure $\mathcal{Q}$ of $\mathcal{M}_\gamma$, with $\alpha \subset \mathcal{Q}$ and $|\mathcal{Q}| = |\alpha|$, such that*

$$\mathrm{ult}(\mathcal{Q}, \pi_{\gamma'}, \tilde{\alpha})$$

is not iterable.

The case hypothesis asserts that the hypothesis of Lemma 7 holds. We define then $\mathcal{M}_\phi$ to be the transitive collapse of $\mathcal{Q}$ as obtained from the lemma. We call this a *special drop* and, as with other drops, we leave the embedding $i_{\nu,\phi}$ undefined for $\nu < \phi$. Also, we call the pair $\{\gamma, \phi\} = \{\gamma, \gamma + 1\}$ a *special pair*.

Case 3. Cases 1 and 2 do not hold.

This part of the definition is completely standard. If either $\mathcal{M}_\gamma$ or $\mathcal{N}_{\gamma'}$ is an initial segment of the other, the iteration stops. Otherwise, let ξ be the least ordinal such that either

1. $\mathcal{U}_\xi^{\mathcal{M}_\gamma} \neq \mathcal{U}_\xi^{\mathcal{N}_{\gamma'}}$, or
2. $\xi \in \mathrm{dom}(\mathcal{U}^{\mathcal{M}_\gamma}) \setminus \mathrm{dom}(\mathcal{U}^{\mathcal{N}_{\gamma'}})$, or
3. $\xi \in \mathrm{dom}(\mathcal{U}^{\mathcal{N}'_\gamma}) \setminus \mathrm{dom}(\mathcal{U}^{\mathcal{M}_\gamma})$.

Then we use $\mathcal{U}_\xi^{\mathcal{M}_\gamma}$ and $\mathcal{U}_\xi^{\mathcal{N}_{\gamma'}}$ to define the models $\mathcal{M}_{\gamma+1}$ and $\mathcal{N}_{\gamma'+1}$ as in the standard comparison. After this, we use the embeddings $\pi_\nu, \nu < \theta$, and the shift lemma of Martin and Steel [Mar94] to define $\widetilde{\mathcal{N}}_{\gamma'+1}$ and $\pi_{\gamma'+1}$.

3.5 The Indiscernibles Generated by $\langle \mathcal{M}_\nu \rangle$

Lemma 8. *The iteration $\langle \mathcal{M}_\nu \rangle$ has length $\delta + 1$.*

Proof. Let $\phi + 1$ be the length of the iteration $\langle \mathcal{M}_\nu \rangle$, and $\theta + 1$ the length of the iteration $\langle \mathcal{N}_\nu \rangle$. The initial models cannot strongly agree up to δ, so, by the proof of the Comparison Lemma, $\phi, \theta \leq \delta$.

Claim 9. The final models $\mathcal{M}_\phi$ and $\mathcal{N}_\theta$ have cardinality at least δ, and hence agree up to δ.

Proof of the claim. We have that the starting models have cardinality at least δ. If one of the final models has cardinality less than δ, in particular the one that is an initial segment of the other has cardinality less than δ, and therefore there is a drop in the corresponding iteration. This is impossible in a standard iteration; we will check that it leads to a contradiction in this case too.

First, assume that there is a standard drop in $\langle \mathcal{M}_\nu \rangle$. Then there is a subset x of the projectum ρ of $\mathcal{M}_{\nu+1}$ which is definable in $\mathcal{M}_{\nu+1}$ but is not a member of the corresponding model in the iteration $\langle \mathcal{N}_\nu \rangle$. If $\nu + 1$ is the last place where a standard drop happened, then $i_{\nu+1,\phi}$ exists and it is the identity when restricted to ρ. Then x is definable in $\mathcal{M}_\phi$, and not a member of $\mathcal{N}_\theta$; therefore, $\mathcal{M}_\phi$ is not an initial segment of $\mathcal{N}_\theta$.

A similar argument works when there are standard drops in $\langle \mathcal{N}_\nu \rangle$. If there are standard drops in both in both iterations, then the argument above shows that the final models for both iterations are equal, and a slightly more complicated argument, looking at the last place at which either of the trees drop, shows that this leads to a contradiction.

Now, assume that there are no standard drops in either iteration. Then, if we assume that there is a special drop in $\langle \mathcal{M}_\nu \rangle$ at the γ-th stage, we have that $\mathcal{R} = \text{ult}(\mathcal{M}_\gamma, \pi_{\gamma'}, \tilde{\alpha})$, is not iterable, where γ' is the corresponding stage in $\langle \mathcal{N}_\nu \rangle$ and $\tilde{\alpha} = \sup \pi``\alpha$ and α is the largest ordinal such that $\mathcal{M}_\gamma$ and $\mathcal{N}_{\gamma'}$ agree up to α. Consequently, $\text{ult}(\mathcal{M}_\phi, \pi_\theta, \tilde{\delta})$ is not iterable. We conclude then that $\mathcal{M}_\phi$ cannot be an initial segment of $\mathcal{N}_\theta$: if it were, we have that $\text{ult}(\mathcal{M}_\phi, \pi_\theta, \tilde{\delta})$ can be embedded into $\text{ult}(\mathcal{N}_\theta, \pi_\theta, \tilde{\delta})$, and that means that $\text{ult}(\mathcal{N}_\theta, \pi_\theta, \tilde{\delta})$ is not iterable. But this ultrapower is embedded in $\widetilde{\mathcal{N}}_\theta$, which is a standard iterate of K. This contradicts the iterability properties of K. □

Now, in order to prove Lemma 8, we notice that if there is any drop in $\langle \mathcal{M}_\nu \rangle$, then the iteration has length δ, since the result of every single step ultrapower has the same cardinality as its predecessor.

So, for the rest of the proof of this lemma, assume that there are no drops in $\langle \mathcal{M}_\nu \rangle$; consequently, $\mathcal{M}_\phi$ is a weasel. If we set $\tilde{\delta} = \bigcup \pi_\theta``\delta$ and $\mathcal{R} = \text{ult}(\mathcal{M}_\phi, \pi_\theta, \tilde{\delta})$ then, by the construction of the iteration $\langle \mathcal{M}_\nu \rangle$ we have that $\mathcal{R}$ is iterable. Therefore, we can compare the models $\mathcal{R}$ and $\widetilde{\mathcal{N}}_\theta$ using iterations $\langle \mathcal{R}_\nu \rangle$ on $\mathcal{R}_0 = \mathcal{R}$ and $\langle \mathcal{S}_\nu \rangle$ on $\widetilde{\mathcal{N}}_\theta$.

Claim 10. 1. There are no drops on either $\langle \mathcal{R}_\nu \rangle$ or $\langle \mathcal{S}_\nu \rangle$.

2. The iterations $\langle \mathcal{R}_\nu \rangle, \langle \mathcal{S}_\nu \rangle$ have the same last model.

Proof. For (1), suppose first that there is a drop in $\langle \mathcal{S}_\nu \rangle$. Notice that, since we are assuming that there are no drops in $\langle \mathcal{M}_\nu \rangle$, there is an elementary embedding from K into $\mathcal{R}$, and therefore $\mathcal{R}$ is universal. Since there is a drop in $\langle \mathcal{S}_\nu \rangle$ then the final model of $\langle \mathcal{R}_\nu \rangle$ is an initial segment of the last model of $\langle \mathcal{S}_\nu \rangle$. But, since this comparison is standard, there can be drops in at most one of the two iterations. Therefore, there are no drops in $\langle \mathcal{R}_\nu \rangle$, and since $\mathcal{R}$ is a proper class, then the final model of $\langle \mathcal{R}_\nu \rangle$

is a proper class. That means that the final model of $\langle \mathcal{S}_\nu \rangle$ is also a proper class, contradicting the assumption that there is a drop in $\langle \mathcal{S}_\nu \rangle$.

Suppose now that there is a drop in $\langle \mathcal{R}_\nu \rangle$. In this case, there are no drops in the iteration $\langle \mathcal{S}_\nu \rangle$ and its last model is an initial segment of the last model of $\langle \mathcal{R}_\nu \rangle$. However, the model $\mathcal{S}_0 = \widetilde{\mathcal{N}}_\theta$ is a proper class, since there are no drops in $\langle \widetilde{\mathcal{N}}_\nu \rangle$: otherwise, there is also a drop in $\langle \mathcal{N}_\nu \rangle$, and that means that $\mathcal{M}_\phi$ is an initial segment of $\mathcal{N}_\theta$, which is impossible because $\mathcal{M}_\phi$ is a proper class. Therefore, there is an elementary embedding from K into the weasel $\mathcal{S}_0 = \widetilde{\mathcal{N}}_\theta$, and we can repeat the argument from the previous paragraph to get a contradiction.

For (2), it is enough to notice that we have embeddings $t = i^{\langle \mathcal{R}_\nu \rangle} \circ \pi_\theta \circ i^{\langle \mathcal{M}_\nu \rangle}$ and $s = i^{\langle \mathcal{S}_\nu \rangle} \circ i^{\langle \widetilde{\mathcal{N}}_\nu \rangle}$ from K into the final models of $\langle \mathcal{R}_\nu \rangle$ and $\langle \mathcal{S}_\nu \rangle$, one of which is an initial segment of the other. By Lemma 2.8(3) from Mitchell [Mit99], we conclude that the two final models are equal, and that $t = s$. This finishes the proof of the claim. $\square$

The following diagram describes the situation:

$$\begin{tikzcd}
 & P & P \\
\mathcal{M}_\phi \arrow[r, "\pi_\theta"] & \mathcal{R} \arrow[u, "i^{\langle\mathcal{R}_\nu\rangle}"'] & \widetilde{\mathcal{N}}_\theta \arrow[u, "i^{\langle\mathcal{S}_\nu\rangle}"'] \\
K \arrow[u, "i^{\langle\mathcal{M}_\nu\rangle}"'] \arrow[uur, bend left, "t"] & & K \arrow[u, "i^{\langle\widetilde{\mathcal{N}}_\nu\rangle}"] \arrow[uu, bend right, "s"']
\end{tikzcd}$$

As we have seen, s and t have the same critical point ρ, $s(\rho) = t(\rho)$ and $t \restriction \mathcal{P}(\rho) = s \restriction \mathcal{P}(\rho)$.

We claim that $s(\rho) \geq \tilde{\delta}$. Suppose, to the contrary, that $s(\rho) < \tilde{\delta}$ and let U be the first ultrafilter used in the iteration $\langle \widetilde{\mathcal{N}}_\nu \rangle$. Then $\rho = \mathrm{crit}(U)$, and $U = \pi_1(\overline{U})$, where $\overline{U}$ is the first ultrafilter used in the iteration $\langle \mathcal{N}_\nu \rangle$. Now let $\xi = \mathrm{crit}(\pi) = \mathrm{crit}(\pi_1) = \mathrm{crit}(\pi_\theta)$; since $t = i^{\langle \mathcal{R}_\nu \rangle} \circ \pi_\theta \circ i^{\langle \mathcal{M}_\nu \rangle}$, we have that $\rho = \mathrm{crit}(t) \leq \xi$. It follows that $\rho < \xi$ since $\rho = \mathrm{crit}(U) = \pi(\mathrm{crit}(\overline{U}))$ and $\xi \notin \mathrm{range}(\pi)$. Now, since

$\rho = \mathrm{crit}(t) < \mathrm{crit}(i^{\langle\mathcal{R}_\nu\rangle} \circ \pi_\theta)$, it follows that $\rho = \mathrm{crit}(i^{\langle\mathcal{M}_\nu\rangle})$, therefore the iteration $\langle\mathcal{M}_\nu\rangle$ starts with an ultrapower using an ultrafilter U' such that $\mathrm{crit}(U') = \rho$. But since $t \restriction \mathcal{P}(\rho) = s \restriction \mathcal{P}(\rho)$, and π is the identity below ξ, it follows that $U' = \overline{U}$; this contradicts the construction of the comparison.

Therefore, $s(\rho) \geq \tilde{\delta}$. Since $i^{\langle\mathcal{R}_\nu\rangle} \circ \pi_\theta \circ i^{\langle\mathcal{M}_\nu\rangle}(\rho) = t(\rho) = s(\rho) \geq \tilde{\delta}$ and $i^{\langle\mathcal{M}_\nu\rangle} \restriction \tilde{\delta}$ is the identity, we have that $\pi_\theta \circ i^{\langle\mathcal{M}_\nu\rangle}(\rho) \geq \tilde{\delta}$; consequently, $i^{\langle\mathcal{M}_\nu\rangle}(\rho) \geq \delta$. This implies the existence of a proper extender, which we have ruled out by assuming that for all cardinals η, $o(\eta) < \eta^{++}$.

This contradiction leads us to conclude that *there is* a drop in $\langle\mathcal{M}_\nu\rangle$.

This finishes the proof of Lemma 8. □

Corollary 11. *There is no drop in* $\langle\mathcal{N}_\nu\rangle$.

The proof is the same as Mitchell's [Mit99].

At this point there are two possibilities:

Case 1. There is an ultrafilter in $\langle\mathcal{N}_\nu\rangle$ that is iterated cofinally many times.

In this case it can be proved, using the proof for Corollary 3.10 in Mitchell [Mit99], that $i^{\langle\mathcal{N}_\nu\rangle}_{0,\theta}``\delta \subset \delta$ and there is a closed and unbounded subset I of δ satisfying the following conditions:

1. If $\nu \in I$ then $\nu = \mathrm{crit}(i^{\langle\mathcal{M}_\nu\rangle}_{\nu,\delta})$.
2. If $\nu, \nu' \in I$ with $\nu < \nu'$ then $\nu' = i^{\langle\mathcal{M}_\nu\rangle}_{\nu,\nu'}(\nu)$.
3. If $\nu \in I$ then $i^{\langle\mathcal{N}_\nu\rangle}_{0,\theta}``\nu \subset \nu$.
4. Every member of I is regular in $\overline{W}$, and is a limit member of $\pi^{-1}(C)$.

Since this set of indiscernibles was constructed using the same ultrafilter, it is a set of true indiscernibles. Therefore, they can be used to find the normal set of indiscernibles for $\mathcal{A}$. The order type of I is δ.

Case 2. For every ultrafilter in $\overline{W}$ there is an ordinal $\beta < \delta$ such that the ultrafilter is iterated less than β times.

In this case, every time an ultrafilter U in the iteration $\langle \mathcal{N}_\nu \rangle$ stops being iterated, it means that the image of U matches an ultrafilter in the iteration $\langle \mathcal{M}_\nu \rangle$. That is, if the comparison process is at the stage corresponding to the models $\mathcal{N}_{\nu_0}$ and $\mathcal{M}_{\nu_1}$ when the image U' of U is no longer used to construct the ultrapowers in the iteration $\langle \mathcal{N}_\nu \rangle$, we have that U' is in both $\mathcal{N}_{\nu_0}$ and $\mathcal{M}_{\nu_1}$. Therefore, the critical point of U' is a measurable cardinal in $\mathcal{N}_{\nu_0}$; since further iterations do not affect U', we have that the critical point of U' is a measurable cardinal in the final model $\mathcal{N}_\theta$. Since there are no drops in the iteration $\langle \mathcal{N}_\nu \rangle$ (again, using the proof of Corollary 3.10 in Mitchell [Mit99]), we have that $\overline{W}$ is elementarily embedded in the final model of the iteration. Hence, since the iterations do not move δ, we have that δ is a limit of measurable cardinals in $\overline{W}$.

Now, $\sup \pi``\delta = \sup(X \cap \kappa)$. Since $C \in X$ and X is an elementary substructure of a segment of V that contains κ, we have that C is closed unbounded in $X \cap \kappa$. This implies that $\sup(X \cap \kappa) \in C$. But by elementarity of π, $\sup(X \cap \kappa)$ is the limit of $\mathrm{cof}(\delta)$ measurable cardinals. Since C was chosen arbitrarily, we obtain that the set of limits of $\mathrm{cof}(\delta)$ measurable cardinals is a stationary subset of κ in K.

This ends the proof of Theorem 4. $\square$

CONCLUSIONS

The three topics considered in this work are quite different, but there is a common theme: the somewhat unusual behavior of well studied objects and concepts under different assumptions about the set theoretical universe.

From the study of finiteness in Chapter 1 we saw that even a concept so intuitively clear as the concept of finite set lends itself to different interpretations when the regulating influence of AC is missing. Although many of the notions of finiteness considered cannot be taken as reasonable definitions, some of the results arising from our study of the principles $C(\mathcal{Q})$ can be interpreted as providing a missing factor that makes some of the alternative notions to be equivalent to the usual definition of finiteness. For example, $C(\mathrm{VII})$, $C(\mathrm{IV})$, $C(\Delta_3)$, and $C(\mathrm{Ia})$ turn out to be equivalent to $E(\mathrm{I},\mathrm{VII})$, $E(\mathrm{I},\mathrm{IV})$, $E(\mathrm{I},\Delta_3)$, and $E(\mathrm{I},\mathrm{Ia})$, respectively. However, this results do not produce new definitions of finiteness equivalent to notion I, since the implications $C(\mathcal{Q}) \longrightarrow E(\mathrm{I},\mathcal{Q})$ use $C(\mathcal{Q})$ as a global property, not a property of a given set (that is, the proofs use the fact that *all* $\mathcal{Q}$-finite families have a choice function).

The unusual topological spaces presented in Chapter 2 show how important is AC in topology. One interesting feature is that the methods used to produce the examples could be used to obtain other countable spaces with interesting properties. For example, we have obtained a model of ZF where there is a countable topological space which is not compact, but in which every infinite subset has a complete accumulation point and every nest of non-empty closed sets has a non-empty intersection. Under AC, both these statements are equivalent to compactness.

Chapter 3 is different from the previous ones because AC no longer plays a relevant role. However, the set theoretical assumptions considered are more interesting, in the

sense that they represent a change in consistency strength. The conclusion obtained —which is just an extension of Mitchell's result, although using more restrictive hypotheses— shows that the rich structure of the core model can be used to bring to the forefront the hidden hypotheses involved in postulates like the existence of Jónsson cardinals.

APPENDIX A
CARDINAL NUMBERS IN SET THEORY WITHOUT CHOICE

In this appendix we offer the definitions and some of the elementary properties of cardinal numbers, their order, and the operations on them. We follow Jech [Jec73]; the reader is referred to the original for a complete exposition of the subject.

A.1 Definition of Cardinal Numbers

The basic notion is that of *equivalence*: two sets are equivalent if there exists a one-to-one function from one onto the other. The underlying idea is that two sets have the same size if and only if they are equivalent.

In order to define cardinal numbers, that is, objects that correspond univocally to each size, we would like to use the equivalence classes themselves. However, each equivalence class turns out to be a proper class; this is quite a limitation, since we cannot formally deal with arbitrary sets of proper classes, and therefore we can at most perform finitary operations on cardinal numbers. The definition avoids this problem by using the division of the set theoretic universe in ranks.

Definition 1. Let x be a set. We define the *cardinal number* (or simply the *cardinal*) of x as the set $|x|$ given by

$$\begin{aligned} |x| = \{y : y &\text{ is a set of minimal rank such that} \\ &\text{there exists a one-to-one map from } x \text{ onto } y\} \end{aligned} \tag{1}$$

This definition also works in ZFA, as long as the class of atoms is a set. If there is a proper class of atoms, then the cardinal numbers defined above can be proper classes, and even more, there is a model of ZFA (constructed as a permutation model,

but having a proper class of atoms) where it is impossible to define cardinal numbers as sets (see Jech [Jec73]).

We remark that in this dissertation we assume that the class of atoms in the models of ZFA is always a set. Therefore, we can use the definition above

A.2 Ordering of Cardinal Numbers

We present here two partial orders on the class of cardinal numbers; these two orders are equivalent under AC.

Definition 2. Let x, y be sets. We say that

$$|x| \leq |y|$$

if there exists a one-to-one map from x into y. We say that

$$|x| \leq^* |y|$$

if there exists a map from y onto x.

The definitions of $\geq$, $\geq^*$, $<$, $>$, $< *$, and $>^*$ can be easily obtained from those above in the usual way.

It is clear that $|x| \leq |y|$ implies $|x| \leq^* |y|$. The converse implication does not hold in general in ZF (and ZFA). The Cantor-Bernstein theorem states that if $|x| \leq |y|$ and $|y| \leq |x|$ then $|x| = |y|$; the same statement is not true in general if we substitute $\leq$ by $\leq^*$.

A.3 Operations on Cardinal Numbers

Finitary operations with cardinal numbers can be always defined; however, infinitary operations require the ability to pick infinitely many representatives, one from each cardinal number involved. We present the definitions below.

Definition 3. Let $\mathfrak{a}, \mathfrak{b}$ be cardinal numbers, and assume that we have disjoint sets a, b such that $|a| = \mathfrak{a}$ and $|b| = \mathfrak{b}$. Then we define:

1. $\mathfrak{a} + \mathfrak{b} = |a \cup b|$.

2. $\mathfrak{a} \cdot \mathfrak{b} = |a \times b|$.

3. $\mathfrak{a}^{\mathfrak{b}} = |{}^{b}a|$.

Notice that $2^{\mathfrak{a}} = |\mathcal{P}(a)|$.

We must remark that if AC fails, the natural candidates to be the definition of $\sum_{i \in I} \mathfrak{a}$ and $\prod_{i \in I} \mathfrak{a}$, that is, $|\bigcup\{a : i \in I\}|$ and $|\prod\{a : i \in I\}|$, respectively, can give ambiguous results, even if a set representatives $\{a_i, i \in I\}$ such that $|a_i| = \mathfrak{a}_i$ for all $i \in I$ can be found.

APPENDIX B
ELEMENTARY CONCEPTS OF PIECEWISE LINEAR TOPOLOGY

In this appendix we offer some of the elementary definitions and results of Piecewise Linear Topology, adapted from the first few chapters from Hudson [Hud69]. The reader is referred to the original for a complete exposition of the subject, although we must mention that we changed some of the definitions. The two lemmas at the end of the appendix are proved here since they do not appear in that book.

B.1 Basic Definitions

We say that the points $r_0, \ldots, r_{k-1}$ in $\mathbb{R}^n$ are *linearly dependent* (in the affine sense) if there exist real numbers $\lambda_0, \ldots, \lambda_{k-1}$, not all zero, such that

$$\sum_{i=0}^{k} \lambda_i r_i = 0 \quad \text{and} \quad \sum_{i=0}^{k} \lambda_i = 0.$$

We say that $r_0, \ldots, r_{k-1}$ in $\mathbb{R}^n$ are *linearly independent* (in the affine sense) if they are not linearly dependent.

An *k-simplex* in $\mathbb{R}^n$ is the convex hull of $k+1$ linearly independent points, called its *vertices*. The convex hull of any subset of of the set of vertices of a k-simplex A is called a *face* of A; each face of a k-simplex is a k'-simplex, with $k' \leq k$. It is clear that every point in a simplex A can be expressed uniquely as a convex combination of the vertices of A.

A *simplicial complex* K is a finite set of simplices such that

1. If $A \in K$ and B is a face of A, then $B \in K$, and
2. If $A, B \in K$, then $A \cap B = \emptyset$ or $A \cap B$ is a face of both A and B.

Given a simplicial complex K, we call the set $||K|| = \bigcup K \subset \mathbb{R}^n$ the *underlying set* of K. Against common usage, we will call $||K||$ a *polyhedron* only if it is a connected subset of $\mathbb{R}^n$; if P is a polyhedron and K is a simplicial complex such that $||K|| = P$, we say that K is a *triangulation* of P.

Given a simplex A, $\widetilde{A}$ denotes the complex whose elements are A and all its faces, $\dot{A}$ denotes the complex whose elements are all the proper faces of A, and $\overset{\circ}{A}$ denotes the set $A \setminus ||\dot{A}||$.

B.2 Subdivisions, Joins, Stars and Links

If K, L are simplicial complexes, K is called a *subdivision* of L if

1. $||K|| = ||L||$, and
2. Every simplex of K is a subset of some simplex of L.

Let A, B be simplices in $\mathbb{R}^n$. If the set of all the vertices of A and of B is linearly independent, then we say that A and B are *joinable*. The simplex whose vertices are those of A and B is called the *join* of A and B, and it is denoted by $A.B$.

We say that two simplicial complexes K, L are *joinable* if

1. If $A \in K$ and $B \in L$, then A, B are joinable; and
2. If $A, A' \in K$ and $B, B' \in L$, then either $A.B \cap A'.B' = \emptyset$ or $A.B \cap A'.B'$ is a face of $A.B$ and of $A'.B'$.

If K, L are joinable, we define

$$K.L = K \cup L \cup \{A.B : A \in K, B \in L\}.$$

Clearly, $K.L$ is a simplicial complex, and we call it the *join* of K and L.

Example. If A, B are joinable simplices, then $\widetilde{A}, \widetilde{B}$ are joinable complexes and $\widetilde{A}.\widetilde{B} = \widetilde{A.B}$.

Now let K be a simplicial complex. If $A \in K$, then we make the following definitions:

1. $\mathrm{star}(A; K) = \{B \in K : A \text{ is a face of } B\}$.

2. $\widetilde{\text{star}}(A;K) = \{B \in K : B \text{ is a face of an element of } \text{star}(A;K)\}$.

3. $\text{link}(A;K) = \{B \in K : B \text{ and } A \text{ are joinable and } A.B \in K\}$.

It can be verified that $\widetilde{\text{star}}(A;K)$ and $\text{link}(A;K)$ are complexes, that $\widetilde{A}$ and $\text{link}(A;K)$ are joinable, and that

$$\widetilde{\text{star}}(A;K) = \widetilde{A}.\, \text{link}(A;K).$$

B.3 Simplicial Maps and Piecewise Linear Maps

If K, L are simplicial complexes, a *simplicial map* $f : K \to L$ is a continuous map $f : ||K|| \to ||L||$ which maps vertices of K to vertices of L and simplices of K linearly into (and therefore onto) simplices of L.

Notice that a simplicial map is determined by its values on the vertices of the domain. Conversely, a function g that assigns to each vertex of a complex K a vertex of a complex L in such a way that if $v_1, \ldots, v_k$ are in a simplex of K then $g(v_1), \ldots, g(v_k)$ are in a simplex of L, then there exists a unique simplicial map f that extends g.

We will define a *piecewise linear map* from a polyhedron P to a polyhedron P' as a map that is a simplicial map with respect to some triangulations K, K' of P and P'.

Lemma 1. *Let K, L be joinable complexes. Suppose that $f\colon ||L|| \to ||L||$ is a piecewise linear homeomorphism. Then there exists a piecewise linear homeomorphism $f'\colon ||K.L|| \to ||K.L||$ that extends f and is constant on $||K||$.*

Proof. Without loss of generality, we can assume that f is a simplicial map from a subdivision L' of L to another subdivision L'' of L. It can be checked that K is joinable to both L' and L'', and that $K.L'$ and $K.L''$ are subdivisions of $K.L$. Let f' be the simplicial map from $K.L'$ to $K.L''$ such that

1. $f'(A) = A$, for all $A \in K$,

2. $f'(B) = f(B)$, for all $B \in L'$, and

3. $f'(A.B) = A.f(B)$, for all $A \in K$ and $B \in L'$.

Then f' satisfies the required conditions. $\square$

Lemma 2. *Let K be a simplicial complex, and let $A \in K$. If f is a piecewise linear homeomorphism from A onto A which is the identity on $\dot{A}$, then f can be extended to a piecewise linear homeomorphism g from $||K||$ onto $||K||$ that moves no points of $||K|| \smallsetminus ||\widetilde{\mathrm{star}}(A;K)||$*

Proof. Since $\widetilde{\mathrm{star}}(A;K) = \widetilde{A}.\operatorname{link}(A;K)$, we can use Lemma 1 to extend f to a piecewise linear homeomorphism f' from $||\widetilde{\mathrm{star}}(A;K)||$ onto $||\widetilde{\mathrm{star}}(A;K)||$ which is the identity on $||\operatorname{link}(A;K)||$.

If $a \in ||K|| \smallsetminus ||\widetilde{\mathrm{star}}(A;K)||$, and $a \in B \in K$, then $C = B \cap ||\widetilde{\mathrm{star}}(A;K)||$ is either empty or a simplex in $\widetilde{\mathrm{star}}(A;K)$. If C contains a point in $\mathring{A}$, then C contains A and then $C \in \widetilde{\mathrm{star}}(A;K)$, which is a contradiction. Therefore, if $C \cap A \neq \emptyset$ then $C \cap A \in \dot{A}$. In that case, $C \in \dot{A}.\operatorname{link}(A;K)$, and since the elements of $\dot{A}$ and of $\operatorname{link}(A;K)$ are not moved by f', we have (by the proof of Lemma 1) that f' is the identity on C. This way, if we define g as the extension of f' to $||K||$ that is the identity on $a \in ||K|| \smallsetminus ||\widetilde{\mathrm{star}}(A;K)||$, we have that g is continuous, and therefore a piecewise linear homeomorphism from $||K||$ onto $||K||$. $\square$

REFERENCES

[Deg94] Degen, J. W. Some aspects and examples of infinity notions. *Math. Logic Quart.*, 40(1):111–124, 1994.

[How98] Howard, Paul and Rubin, Jean E. *Consequences of the axiom of choice.* American Mathematical Society, Providence, RI, 1998. With 1 IBM-PC floppy disk (3.5 inch; WD).

[How94] Howard, Paul and Spišiak, Ladislav. Definitions of finite and the power set operation. Preprint, 1994.

[How89] Howard, Paul and Yorke, Mary F. Definitions of finite. *Fund. Math.*, 133(3):169–177, 1989.

[Hud69] Hudson, J. F. P. *Piecewise linear topology.* W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[Jec73] Jech, Thomas J. *The axiom of choice.* North-Holland Publishing Co., Amsterdam, 1973. Studies in Logic and the Foundations of Mathematics, Vol. 75.

[Jec78] Jech, Thomas. *Set theory.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[Jec66] Jech, T. and Sochor, A. Applications of the θ-model. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 14:351–355, 1966.

[Lév58] Lévy, A. The independence of various definitions of finiteness. *Fund. Math.*, 46:1–13, 1958.

[Mar94] Martin, D. A. and Steel, J. R. Iteration trees. *J. Amer. Math. Soc.*, 7(1):1–73, 1994.

[Mil93] Miller, Arnold W. Arnie Miller's problem list. In *Set theory of the reals (Ramat Gan, 1991)*, pages 645–654. Bar-Ilan Univ., Ramat Gan, Israel, 1993. Updated version, preprint.

[Mit99] Mitchell, W. J. Jónsson cardinals, Erdős cardinals, and the core model. *J. Symbolic Logic*, 64(3):1065–1086, Sept. 1999.

[Mos38] Mostowski, Andrzej. Über den Begriff einer endlichen Menge. *C. R. des Séances de la Société des Sciences et des Lettres de Varsovie*, 31(Cl. III):13–20, 1938.

[Pin72] Pincus, D. Zermelo–Fraenkel consistency results by Fraenkel–Mostowski methods. *J. Symbolic Logic*, 37(4):721–743, Dec. 1972.

[Sie52] Sierpinski, Waclaw. *General topology.* University of Toronto Press, Toronto, 1952.

[Spi93] Spišiak, Ladislav. Dependences between definitions of finiteness. II. *Czechoslovak Math. J.*, 43(118)(3):391–407, 1993.

[Spi88] Spišiak, Ladislav and Vojtáš, Peter. Dependences between definitions of finiteness. *Czechoslovak Math. J.*, 38(113)(3):389–397, 1988.

[Tar24] Tarski, A. Sur les ensembles finis. *Fund. Math.*, 6:45–95, 1924.

[Tru74] Truss, John. Classes of Dedekind finite cardinals. *Fund. Math.*, 84(3):187–208, 1974.

BIOGRAPHICAL SKETCH

I was born in Barquisimeto, Venezuela, in 1970 and I lived in nearby Quíbor until age 18. My bachelor's degree in mathematics was granted by the Universidad Centrooccidental "Lisandro Alvarado" in Barquisimeto. Afterwards, I obtained a master's degree in mathematics at the Venezuelan Institute for Scientific Research (IVIC) in Caracas, under the guidance of Dr. Carlos A. Di Prisco.

In 1995 I started graduate studies in mathematics at the University of Florida, under the guidance of Dr. W. J. Mitchell, and worked for the Department of Mathematics as a teaching assistant. Currently, I have accepted a position as a visiting assistant professor at Purdue University for the years 2000–2001 and 2001–2002.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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Dean, Graduate School

THREE TOPICS IN SET THEORY:
FINITENESS AND CHOICE, CARDINALITY OF COMPACT SPACES,
AND SINGULAR JÓNSSON CARDINALS

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Set theory is the area of mathematics that provides the foundations for most of the work in all areas of mathematics. It is also a very active field of research by itself.

In this dissertation we explore how some basic assumptions about the set theoretic universe affect set theory and topology. The first assumption studied is the axiom of choice (AC); it is well known that if AC fails in the universe of sets, then the distinction between infinite and finite becomes slightly blurry, and we find in turn how this affects AC. Also, the failure of AC allows us to find pathological examples in topology, the area of mathematics that studies continuity from the geometric and analytic point of view.

A different kind of assumption that we study is the existence of so–called large cardinal numbers, which measure the size of different large infinite sets. The existence of large cardinals is an assumption that, unlike AC, cannot be proved consistent with the other axioms of set theory. We use core model theory, a cutting edge area of research in set theory, to study the properties of a kind of large cardinals called *Jónsson cardinals.*